A Result for Semi Simple Ring

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Abstract

If for a semi simple ring R, with any elements a, b in R there exist positive integers $p = p(a, b)$ and $q = q(a, b)$ such that $[(bab)^p, (ab)^q + (ba)^q] = 0$. then R is commutative.

Key words - *Skew field, Simple ring, Semi simple ring & Primitive ring.*

I. INTRODUCTION

M. Ashraf [1] proved: Let Rbe a semi simple ring. Suppose that given x, yin R there exist positive integers $m = m(x, y)$and $n = n(x, y)$such that $[x^m, (xy)^n] = [(yx)^n, x^m]$ then $R$is commutative.
We have weakened the M. Ashraf identity to make a stronger generalization of M. Ashraf [1]. This reads as follows
:

A. THEOREM: Let Rbe a semi simple ring. Suppose that given x, yin R, there exist positive integers $p = p(a, b)$and $q = q(a, b)$such that $[(\boldsymbol{bab})^{\boldsymbol{p}}, (ab)^q + (ba)^q] = 0$. Then $R$is commutative.
Throughout this paper $R$is taken as an associative ring. $[x, y] = xy - yx$ and $xoy = xy + yx$for every pair x, yin R.

B. Division Ring (Skew Fields)

A division ring, also called a skew field, is a non commutative ring with unity in which every non zero element has inverse.

C. Simple Ring

A simple ring is a non-zero ring that has no two-sided ideal besides the zero ideal and itself.
e.g: Any quotient of a ring by a maximal ideal is a simple ring. In particular, a field is a simple ring. In fact a division ring is also a simple ring.

D. Semi Simple Rings

A ring in which unit element does not equal to zero element and which is semi simple as (left) module over itself is semisimple ring.

E. Primitive Rings

A ring R is said to be a left primitive ring if and only if it has a simple left R-module. A right primitive ring is defined similarly with right R-modules.

II. PREPARATORY RESULTS

This section contains those lemmas which are required to prove the theorem.

LEMMA 2.1: ([3], Theorem 1) : Let Rbe a ring without nonzero nil right ideals. Suppose that given a, bin R. there exist positive integers $p = p(x, y)$, $q = q(x, y) > 1$ and $t = t(x, y) > 1$ such that $[x^p, [x^q, y^t]] = 0$. Then R is commutative.

LEMMA 2.2*:* Let x, y, zbe the elements of a ring R.
(i) If $[x, y] = 0$then $[x, yoz] = yo[x, z]$.
(ii) $xo[x, y] = [x^2, y]$.
PROOF: Proof of this lemma is straight forward.

LEMMA 23: Let $\boldsymbol{R}$ be a Skew field. Suppose that given $\boldsymbol{a}, \boldsymbol{b}$in $\boldsymbol{R}$,there exist positive integers
$\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b})$ and $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{a}, \boldsymbol{b})$such that $[(\boldsymbol{bab})^{\boldsymbol{p}}, (\boldsymbol{ab})^{\boldsymbol{q}} + (\boldsymbol{ba})^{\boldsymbol{q}}] = \boldsymbol{0}$. Then $\boldsymbol{R}$ is commutative.

PROOF: Let $\boldsymbol{b} \neq 0$, then by hypothesis there exist positive integers$\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{b}^{-1}\boldsymbol{a}\boldsymbol{b}^{-1}, \boldsymbol{b})$and $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{b}^{-1}\boldsymbol{a}\boldsymbol{b}^{-1}, \boldsymbol{b})$ such that the identity given in the hypothesis reduces to
$[(b\boldsymbol{b}^{-1}\boldsymbol{a}\boldsymbol{b}^{-1}\boldsymbol{b})^{p}, (\boldsymbol{b}^{-1}\boldsymbol{a}\boldsymbol{b}^{-1}\boldsymbol{b})^{q} + (\boldsymbol{b}\boldsymbol{b}^{-1}\boldsymbol{a}\boldsymbol{b}^{-1})^{q}] = \mathbf{0}$
i.e. $[\boldsymbol{a}^{p}, (\boldsymbol{b}^{-1}\,\boldsymbol{a})^{q} + (\boldsymbol{a}\boldsymbol{b}^{-1})^{q}] = 0$ (1)
Replacing$\boldsymbol{b}$by $\boldsymbol{a}\boldsymbol{b}^{-1}$in (1).we get$[\boldsymbol{a}^{p}, \boldsymbol{b}^{q} + \boldsymbol{a}\boldsymbol{b}^{q}\boldsymbol{a}^{-1}\,] = \mathbf{0}$.
This on simplification and consequently multiplying by $\boldsymbol{a}$ on the right hand side yields,
$\boldsymbol{a}^{p}\boldsymbol{b}^{q} + \boldsymbol{a}^{p+1}\boldsymbol{b}^{q} - \boldsymbol{b}^{q}\boldsymbol{a}^{p+1} - \boldsymbol{a}\boldsymbol{b}^{q}\boldsymbol{a}^{p} = 0$ (2)
Therefore $[\boldsymbol{a}^{p}, \boldsymbol{a}\boldsymbol{o}\boldsymbol{b}^{q}] = 0$Now by lemma 2.2(i), we obtain
$\boldsymbol{a}\boldsymbol{o}[\boldsymbol{a}^{p}, \boldsymbol{b}^{q}]$when$\left[\boldsymbol{a}, \boldsymbol{a}\boldsymbol{o}[\boldsymbol{a}^{p}, \boldsymbol{b}^{q}]\right] = \mathbf{0}$Now using parts (i) and (ii) consecutively of lemma 2.2, we have
$\left[\boldsymbol{a}^{2}, [\boldsymbol{a}^{p}, \boldsymbol{b}^{q}]\right] = \mathbf{0}$.Hence by lemma 2.1, $\boldsymbol{R}$is commutative.

III. PROOF OF THE THEOREM

Suppose that $\boldsymbol{R}$is a semi simple ring such that for all $\boldsymbol{x}, \boldsymbol{y}$in $\boldsymbol{R}$there exist positive integers $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b})$ and $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{a}, \boldsymbol{b})$ fo which $[(\boldsymbol{bab})^{p}, (\boldsymbol{ab})^{q} + (\boldsymbol{ba})^{q}] = 0$ (A)
A semi simple ring is isomorphic to a sub direct sum of primitive rings. Further the identity (A) satisfied by a ring is also satisfied by all its sub rings and homomorphic images. So to prove the theorem for semi simple rings it suffices to prove it for primitive rings. Now every Skew field is primitive (5.20 Ex. 5, [4]), but a primitive ring need not be a Skew field, in general. Here in the present case, the primitive ring satisfying the identity (A) is necessarily a Skew field. Because a primitive ring by Jacobson Density Theorem is isomorphic to a complete matrix ring $\boldsymbol{D}_{\boldsymbol{t}}$where D is a Skew field and $\boldsymbol{t} > 1$. But we observe that identity (A) is not satisfied by any complete matrix ring $\boldsymbol{D}_{\boldsymbol{t}}$e.g. a consideration of $\boldsymbol{a} = \boldsymbol{e}_{\mathbf{11}} + \boldsymbol{e}_{\mathbf{12}}$and $\boldsymbol{b} = \boldsymbol{e}_{\mathbf{11}}$for $\boldsymbol{t} = 2$ leads that (A) is notsatisfied. Thus our primitive ring $\boldsymbol{R}$ is necessarily a Skew field. Hence theorem follows from lemma 2.3.

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