A Class of Exact Solutions of Equations for Plane Steady Motion of Incompressible Fluids of Variable Viscosity for finite Peclet Number through Von-Mises Coordinates

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Abstract

This paper determines a class of exact solutions for plane steady motion of incompressible fluids of variable viscosity for finite Peclet number through von-Mises coordinates. The class is characterized by an equation involving a stream function ψ and two differentiable functions $f(x)$ and $g(x)$. Successive transformations technique is used on non-dimensional form of basic equations. The exact solutions are determined basing on two velocity profile cases. The first velocity profile case fixes the functions $g(x)$ and demands $f(x)$ to satisfy a second order variable coefficients differential equation whose trivial solution is opted. The second velocity profile case fixes only the function $g(x)$ and leaves $f(x)$ arbitrary. In both the cases, a large set of expressions for streamlines, viscosity function, generalized energy function and temperature distribution for finite Peclet number can be found.

Keywords - *Exact solutions for incompressible fluids, Variable viscosity fluids, Navier-Stokes equations with body force, Martin's coordinates, von-Mises coordinates*

I. INTRODUCTION

The basic equations for motion of a fluid element comprises of the equation of momentum, the equation of energy and equation of continuity. The basic equations for plane steady motion of incompressible variable viscosity fluid in Cartesian space (x, y) in non-dimensional form are following

$$u_x + v_y = 0 \tag{1}$$

$$u u_x + v u_y = -p_x + [(2\mu u_x)_x + \{\mu(u_y + v_x)\}_y] \tag{2}$$

$$u v_x + v v_y = -p_y + \frac{1}{R_e}[(2\mu v_y)_y + \{\mu(u_y + v_x)\}_x] \tag{3}$$

$$u T_x + v T_y = \frac{1}{R_e P_r}(T_{xx} + T_{yy}) + \frac{E_c}{R_e}[2\mu(u_x^{\;2} + v_y^{\;2}) + \mu(u_y + v_x)^2] \tag{4}$$

Where the coefficient of viscosity is $\mu > 0$, the velocity vector field $\mathbf{q} = (u(x, y), v(x, y))$ and $p = p(x, y)$ is pressure. The dimensionless quantities R_e, P_r and E_c are respectively the *Reynolds number*, the *Prandtl number* and the *Eckert number*. The product of R_e and P_r is Peclet number $P_{e'}$.
The equation of contunity (1) indicates

$$\psi_y = u, \qquad \psi_x = -v \tag{5}$$

where $\psi = \psi(x, y)$ is a stream function such that $\psi_{yx} = \psi_{xy}$.
Dimension analysis method, coordinates transformation techniques and successive coordinate's transformation techniques are available for exact solutions in references [1-9] and references therein. The solution of equation (4) for very large and very small $P_{e'}$ can be found where as finding solutions for finite $P_{e'}$ is fascinating [10-12]. The successive transformation technique transforms basic equations from Cartesian system (x, y) to Martin's system (φ, ψ) and then to von-Mises system (x, ψ). The Martin's coordinates system (φ, ψ)

defines the curves $\psi = const.$ as streamlines and leaves the curves $\varphi = const.$ arbitrary [13]. With this definition the curvilinear coordinates system (φ, ψ) is called the Martin's coordinates system. In Martin's system the curves $\varphi = const.$ are arbitrary therefore von-Mises coordinates system (x, ψ) takes it along $x - axis$ [14].

Let us characterization the streamlines $\psi = const.$ by

$$\frac{y - f(x)}{g(x)} = const. \tag{6}$$

The equation (7), without loss of generality, implies

$$y = f(x) + g(x)\nu(\psi) \tag{7}$$

where $f(x)$ and $g(x)$ are differentiable functions. In this communication, the function $g'(x)$ is non-zero and $\nu''(\psi)$ is zero.

The paper is organized as follow: Section (2) applies successive transformation technique converts the basic to Martin's coordinates (φ, ψ) then to the von-Mises coordinates (x, ψ). Section (3) finds exact solutions of fundamental equations. The last section presents conclusion.

II. BASIC EQUATIONS IN VON-MISES COORDINATES

In equations (2-4) let us define the vorticity function w and the total energy function L, the functions A and B as follow

$$w = v_x - u_y \tag{8}$$

$$L = p + \frac{1}{2}(u^2 + v^2) - \frac{2\mu u_x}{\mathrm{R}_e} \tag{9}$$

and

$$A = \mu(u_y + v_x), \qquad B = 4\mu u_x \tag{10}$$

Consider the allowable change of Martin's coordinates (φ, ψ) through

$$x = x(\varphi, \psi), \qquad y = y(\varphi, \psi) \tag{11}$$

such that the Jacobian of the transformation $J = \dfrac{\partial(x, y)}{\partial(\varphi, \psi)} \neq 0$ is finite.

Suppose θ be the angle between the tangents to the streamlines lines $\psi = const.$ and the curves $\varphi = const.$ at a common point $P(x, y)$, then

$$\tan(\theta) = \frac{y_\varphi}{x_\varphi} \tag{12}$$

Applying the differential geometric technique [15], the fundamental equations (2-4) in Martin's coordinates system (φ, ψ) are following

$$\begin{aligned} -R_e w J E = {} & R_e J E L_\psi + A_\varphi\left((F^2 - J^2)\cos 2\theta - 2FJ\sin 2\theta\right) \\ & + EA_\psi\left(J\sin 2\theta - F\cos 2\theta)\right) - B_\varphi\left(\frac{1}{2}(F^2 - J^2)\sin 2\theta + FJ\cos 2\theta\right) \\ & + E B_\psi\left(\frac{1}{2}F\sin 2\theta + J\cos^2\theta\right), \end{aligned} \tag{13}$$

$$\begin{aligned} 0 = {} & -R_e J L_\varphi + E A_\psi \cos 2\theta - A_\varphi\left[F\cos 2\theta - J\sin 2\theta\right] \\ & + B_\varphi\left(\frac{1}{2}F\sin 2\theta - J\sin^2\theta\right) - \frac{E B_\psi}{2}\sin 2\theta, \end{aligned} \tag{14}$$

and

$$\frac{1}{J\,P_{e'}}\left[\left(\frac{G\,T_{\varphi}-F\,T_{\psi}}{J}\right)_{\varphi}+\left(\frac{E\,T_{\psi}-F\,T_{\varphi}}{J}\right)_{\psi}\right]=-\frac{\mathrm{E}_{c}}{\mathrm{R}_{e}}\frac{1}{4\mu}\left(B^{2}+4A^{2}\right)+\frac{T_{\varphi}}{J} \tag{15}$$

where the coefficients of first fundamental form are

$$E=x_{\varphi}^{2}+y_{\varphi}^{2}\;,\;F=x_{\varphi}\,x_{\psi}+y_{\varphi}\,y_{\psi}\;,\;G=(x_{\psi})^{2}+(y_{\psi})^{2}, \tag{16}$$

$$J=\pm\sqrt{E\,G-F^{2}}\quad, \tag{17}$$

and

$$A(\varphi,\psi)=\mu\,[-\frac{(F\cos\theta-J\sin\theta)}{4E^{2}J^{5}}\{E_{\varphi}(2EJ^{3}\cos\theta+F\sqrt{E}\sin\theta)$$
$$-4E^{2}J^{2}J_{\varphi}\cos\theta-2E\sqrt{E}\,F_{\varphi}\sin\theta+E\sqrt{E}\,E_{\psi}\sin\theta\}$$
$$+\frac{\cos\theta}{2J^{3}}\{E_{\psi}(F\sin\theta+J\cos\theta)-2EJ_{\psi}\cos\theta-E\,G_{\varphi}\sin\theta\}$$
$$+\frac{(F\sin\theta+J\cos\theta)}{2EJ^{3}}\{(J\,E_{\varphi}-2EJ_{\varphi})\sin\theta$$
$$+\cos\theta\,[-FE_{\varphi}+2E\,F_{\varphi}-E\,E_{\psi}]\}$$
$$-\frac{\sin\theta}{2J^{3}}\{(E_{\psi}(J\sin\theta-F\cos\theta)-2EJ_{\psi}\sin\theta+EG_{\varphi}\cos\theta\}], \tag{18}$$

$$B(\varphi,\psi)=\frac{4\mu}{EJ^{3}}[E_{\varphi}(F\sin\theta+J\cos\theta)^{2}-2E(F\sin\theta+J\cos\theta)$$
$$(F_{\varphi}\sin\theta+J_{\varphi}\cos\theta)+E^{2}(J_{\psi}\sin2\theta+G_{\varphi}\sin^{2}\theta)], \tag{19}$$

and

$$w=\frac{(F\sin\theta+J\cos\theta)}{2EJ^{3}}\{(J\,E_{\varphi}-2EJ_{\varphi})\sin\theta+\cos\theta\,[-FE_{\varphi}+2E\,F_{\varphi}-E\,E_{\psi}]\}$$
$$-\frac{\sin\theta}{2J^{3}}\{E_{\psi}(J\sin\theta-F\cos\theta)-2EJ_{\psi}\sin\theta+EG_{\varphi}\cos\theta\}]$$
$$+\frac{(F\cos\theta-J\sin\theta)}{4E^{2}J^{5}}\{E_{\varphi}(2EJ^{3}\cos\theta+F\sqrt{E}\sin\theta)$$
$$-4E^{2}J^{2}J_{\varphi}\cos\theta-2E\sqrt{E}\,F_{\varphi}\sin\theta+E\sqrt{E}\,E_{\psi}\sin\theta\}$$
$$-[\frac{\cos\theta}{2J^{3}}\{E_{\psi}(F\sin\theta+J\cos\theta)-2EJ_{\psi}\cos\theta-E\,G_{\varphi}\sin\theta\}], \tag{20}$$

Since the von-Mises coordinates system (x,ψ), takes the curves $\varphi=const$. along $x-axis$, therefore

$$\varphi=x \tag{21}$$

Applying equation (21) and the streamlines (7) in equation (12) and equations (16-17) and using the trigonometric identities, we have

$$\cos\theta=\frac{1}{\sqrt{E}} \tag{22}$$

$$E=1+(M+N\,v)^{2} \tag{23}$$

$$F=J\sqrt{E-1} \tag{24}$$

$$G=J^{2} \tag{25}$$

$$J=x\,g\,v'=a\,(x\,g) \tag{26}$$

where

$$N(x) = x\,g'(x) \qquad M(x) = x\,f'(x), \tag{27}$$

and

$$v = a\psi + b \tag{28}$$

with constants $a \neq 0$, b. The equations (13-15) and equations (18-20), in von-Mises coordinates (x,ψ) are following

$$-R_e\,w = R_e\,L_\psi - J\,A_x + \sqrt{E-1}\,A_\psi + B_\psi \tag{29}$$

$$0 = -R_e\,L_x + \frac{A_\psi\,(2-E)}{J} + A_x\sqrt{E-1} - \frac{\sqrt{E-1}\,B_\psi}{J} \tag{30}$$

$$J\,T_{xx} - 2a\sqrt{E-1}\,T_{vx} + \frac{a^2 E}{J}T_{vv} + \left(J_x - \frac{E_\psi}{2\sqrt{E-1}} - P_{e'}\right)T_x$$

$$+a\left(\frac{E_\psi}{J} - \frac{E_x}{2\sqrt{E-1}} - \frac{EJ_\psi}{J^2}\right)T_v = -\frac{J\,E_c\,P_r}{4\mu}\left(B^2 + 4A^2\right) \tag{31}$$

$$A = \frac{\mu}{a\,(x\,g)^2}\left[x\,g\,(M' + N'v) - 2(M + Nv)(N + g)\right] \tag{32}$$

$$B = \frac{-4\mu(N+g)}{a\,x^2 g^2} \tag{33}$$

$$w = \left[\frac{M'}{x\,g} - \frac{2MN}{(x\,g)^2}\right]\left(\frac{1}{a}\right) + \left[\frac{N'}{x\,g} - \frac{2N^2}{(x\,g)^2}\right]\left(\frac{v}{a}\right) \tag{34}$$

and the magnitude of velocity vector $\mathbf{q} = (u, v)$ is

$$q = \frac{\sqrt{1 + (M + Nv)^2}}{J} \tag{35}$$

Eliminating the function L from equations (28-29) assuming $L_{x\psi} = L_{\psi x}$ provides the following compatibility equation

$$a\,x\,g\,A_{xx} - 2x\,(f' + g'v)\,A_{x\psi} - \frac{\left[1 - x^2(f' + g'v)^2\right]}{a\,x\,g}A_{\psi\psi}$$

$$a\,g\,A_x - A_\psi\left((f' + g'v) + x(f'' + g''v)\right) - A_\psi\left((f' + g'v) + x(f'' + g''v)\right)$$

$$-\left\{B_x - \frac{(f' + g'v)\,B_\psi}{a\,g}\right\}_\psi = R_e\,w_x \tag{36}$$

Finding solution of equation (36) L and T are obtained from equations (29-30) and (31) respectively. The viscosity is obtained either from equation (32) or (33). The pressure and velocity components are obtained from equation (9) and equation (5) respectively.

The number of unknown in equation (36) can be reduced by eliminating μ through a relation between A and B but it is not impossible here. Therefore, following two cases are considered

Case I: $\qquad A = 0$

Case II: $\qquad B = 0$

III. EXACT SOLUTIONS

The case

$$A = 0 \tag{37}$$

and equations (27) in (32) provides

$$\frac{2(xg)'g'}{g} - (xg')' = 0 \tag{38}$$

$$\frac{-2(xg)'f'}{g} + (xf')' = 0 \tag{39}$$

It is easy to see that equation (38) is satisfied by

$$g(x) = \frac{-1}{C_0 x^2 + C_1} \tag{40}$$

Equation (39) on utilizing equation (40) provides

$$x(C_0 x^2 + C_1) f'' + (3C_0 x^2 - C_1) f' = 0 \tag{41}$$

The equation (41) possesses trivial and non-trivial solution. A trivial solution of equation (41) is

$$f(x) = 0 \tag{42}$$

Equation (36) on utilizing equation (42), becomes

$$B_{x\nu} - \left(\frac{g'}{g}\right)\nu B_{\nu\nu} - \left(\frac{g'}{g}\right) B_\nu = \frac{-4R_e C_0 \nu}{a^2} g' \tag{43}$$

This suggests seeking a solution of the form

$$B = \nu^2 Q(x) \tag{44}$$

Equation (43) on substituting equation (44) provides

$$Q' - 2\left(\frac{g'}{g}\right) Q = \frac{-2R_e C_0}{a^2} g' \tag{45}$$

whose solution is

$$Q(x) = \frac{2R_e C_0}{a^2} g + C_2 g^2 \tag{46}$$

Equation (44) on substituting equation (46) gives

$$B = \nu^2 \{\frac{2R_e C_0}{a^2} g + C_2 g^2\} \tag{47}$$

and equation (33) on using equation (47) provides

$$\mu = \frac{-a(xg)^2}{4(xg)'}\{\frac{2R_e C_0}{a^2} g + C_2 g^2\}\nu^2 \tag{48}$$

Solution of equation (29) and (30), utilizing equation (47) and (37) is

$$a R_e L = \{-\frac{R_e}{a}\left[\frac{N'}{xg} - \frac{2N^2}{(xg)^2}\right] - 2a\ Q(x)\}\left(\frac{\nu^2}{2}\right) + p_6 \tag{49}$$

Equation (31), using (42) and (47) provides

$$(axg)T_{xx} - 2a(xg'\nu)T_{\nu x} + \frac{a\left[1+(xg'\nu)^2\right]}{xg}T_{\nu\nu} + \left((axg)_x - \frac{a\left[1+(xg'\nu)^2\right]_\nu}{2(xg'\nu)} - P_{e'}\right)T_x$$

$$+a\left[\frac{[2xg'(xg'\nu)]}{(xg)} - (xg')'\nu\right]T_\nu = E_c P_r a^2 (xg)(xg'+g)\nu^2 Q(x) \tag{50}$$

Let us seek solution of equation (50) of the form

$$T(x,\nu) = \nu^2 R(x) + S(x) \tag{51}$$

Equation (50) on inserting equation (51) provides

$$a x^2 g^2 R'' + x g\{a g - P_{e'} - 4a\ x g'\}R' + 2a\{3(x g')^2 - x g (x g')'\}R$$
$$= E_c P_r\ (x g' + g)\{\frac{2 R_e C_0}{a^2} g + C_2 g^2\} \tag{52}$$

$$a\ x^2 g^2\ S'' + x g\{a g - P_{e'}\}S' = -2 a R(x) \tag{53}$$

On substituting the value of $g(x)$ from equation (40) in equation (52) we get

$$a x^2 (C_0 x^2 + C_1)^2 R'' + x(C_0 x^2 + C_1)\{a(9 C_0 x^2 + C_1) + P_{e'}(C_0 x^2 + C_1)^2\}R'$$
$$+8a C_0 x^2 (2 C_0 x^2 + C_1)R = -E_c P_r (C_0 x^2 + C_1)\{\frac{2 R_e C_0^2}{a^2} x^2 + \frac{2 R_e C_0 C_1}{a^2} - C_2\} \tag{54}$$

where C_0, C_1 and C_2 are arbitrary constants. Equation (54) suggests searching a solution of (54) setting

$$C_1 = 0 \tag{55}$$

Equation (54) on substituting $C_1 = 0$, becomes

$$a x^2 R'' + x\{9 a + P_{e'} C_0 x^2\}R' - 16 a R = -E_c P_r\{\frac{2 R_e C_0^2}{a^2} - \frac{C_2}{C_0 x^2}\} \tag{56}$$

Consider the function $R(x)$ satisfying equation (56) takes the form

$$R(x) = B_1 + B_2 x^{-2} \tag{57}$$

implies

$$B_1 = \frac{E_c P_r R_e}{8 a^2}\left(\frac{1}{a} + \frac{P_r C_2}{28}\right) \tag{58}$$

and

$$B_2 = \frac{-E_c P_r C_2}{28\ a C_0} \tag{59}$$

Inserting equations (58-59) in equation (57), we get

$$R(r) = \frac{E_c P_{e'}}{8 a^2}\left(\frac{1}{a} + \frac{P_r C_2}{28}\right) - \frac{E_c P_r C_2}{28 a C_0}\left(\frac{1}{x^2}\right) \tag{60}$$

Employing equation (60) in equation (53), we get

$$S(x) = -2 C_0^2 \int\left[\frac{e^{\frac{-C_0 P_{e'}}{2a} x^2}}{x} \int x^3 e^{\frac{C_0 P_{e'}}{2a} x^2} R(x)\,dx\right]dx + C_3 \int \frac{e^{\frac{-C_0 P_{e'}}{2a} x^2}}{x} dx + C_4 \tag{61}$$

On substituting equations (60-61)) in equation (51), we get the temperature distribution.

Similarly, for non-trivial solution of equation (41), the exact the solutions to the basic equations can be obtained.

The case

$$B = 0 \tag{62}$$

in equation (33) implies

$$g = \frac{c}{x} \tag{63}$$

where $c \neq 0$ is constant.

Equation (36) on supplying $B = 0$ gives

$$a_1 c A_{xx} - 2a\ (M - g v) A_{xv} - \frac{a\left[1 - (M - g v)^2\right]}{c} A_{vv} + a g A_x$$

$$+aA_{\nu}\left(-(M'-g'\nu)+\frac{2(-g)(M-g\nu)}{c}\right)=\left(\frac{R_e}{ac}\right)\left[M'+\frac{2M}{x}\right]'+\frac{2R_e\nu}{ax^3} \tag{64}$$

Consider the solution of equation (64) of the form

$$A=R(x)+P(x)\nu \tag{65}$$

Equation (64) employing equation (65) become

$$acR''-2aMP'+agR'+aP\left(-M'-\frac{2gM}{c}\right)$$

$$+\nu\ \{acP''+3agP'+aP\left(g'+\frac{2g^2}{c}\right)\}=\left(\frac{R_e}{ac}\right)\left[M'+\frac{2M}{x}\right]'+\frac{2R_e\nu}{ax^3} \tag{66}$$

Since ν and x are independent variables therefore the equation (66) yields

$$x^2P''+3xP'+P=-\frac{2R_e}{a^2cx} \tag{67}$$

and

$$(xR')'=\left(\frac{xR_e}{a^2c^2}\right)\left[M'+\frac{2M}{x}\right]'+\frac{2xM}{c}P'+\frac{x}{c}\left(M'+\frac{2M}{x}\right)P \tag{68}$$

The solutions of equations (67-68) are

$$P(x)=\frac{s_1}{x}+\frac{s_2\ln x}{x}-\frac{R_e(\ln x)^2}{a^2cx} \tag{69}$$

and

$$R(x)=\frac{1}{x}\int\left\{\frac{1}{x}\int Z_1(x)\,dx\right\}dx+C_1\ln x+C_2 \tag{70}$$

where

$$Z_1(x)=\left(\frac{xR_e}{a^2c^2}\right)\left[M'+\frac{2M}{x}\right]'+\frac{2xM}{c}P'+\frac{x}{c}\left(M'+\frac{2M}{x}\right)P \tag{71}$$

On substituting equations (69-71) in equation (32), we get the value of μ

$$\mu=\frac{ac}{(M'-g'\nu)}\left(R(x)+P(x)\nu\right) \tag{72}$$

The solution of equation (29) and (30) using equation (63), is

$$aR_eL=\left(\frac{-R_e}{ax^2}+acP'(x)+agP(x)\right)\left(\frac{\nu^2}{2}\right)+\{\left(\frac{-R_e}{ac}\right)\left[M'(x)+\frac{2M(x)}{x}\right]$$

$$+acR'(x)-aM(x)P(x)\}\nu+\frac{a}{c}\int\left(1-M^2\right)P(x)\,dx+\int M(x)R'(x)\,dx+p_5 \tag{73}$$

The equation (31) on utilizing equation (72) becomes

$$acT_{xx}-2a(M-g\nu)T_{\nu x}+\frac{a\left[1+(M-g\nu)^2\right]}{c}T_{\nu\nu}$$

$$+\left(\frac{ac}{x}-P_{e'}\right)T_x-\left(\frac{2(M+N\nu)}{x}(M'-g'\nu)\right)aT_{\nu}$$

$$=-E_cP_r\left[RM'+(PM'-g'R)\nu-Pg'\nu^2\right] \tag{74}$$

The structure of equation (74) suggests

$$T=R_1(x)+R_2(x)\nu+R_3(x)\nu^2 \tag{75}$$

and

$$\nu^2 [a\, c R_3'' + 4 ag\, R_3' + \frac{2 aR_3 g^2}{c} + 2 aR_3 (g' + \frac{2g}{x}) + \left(\frac{ac}{x} - P_{e'}\right) R_3']$$

$$+\nu\, [a\, c R_2'' - 4 aM\, R_3' \nu + 2 ag\, \nu R_2' - \frac{4 aR_3 Mg\, \nu}{c}$$

$$+ \left(\frac{ac}{x} - P_{e'}\right) R_2' \nu + a \left\{ - 2 R_3 \left(M' + \frac{2M}{x}\right) + R_2 \left(g' + \frac{2g}{x}\right)\right\}]$$

$$+ a\, c R_1'' - 2 aM\, R_2' + \frac{2 aR_3 (1 + M^2)}{c} + \left(\frac{ac}{x} - P_{e'}\right) R_1' + \left(- a(M' + \frac{2M}{x}) R_2\right)$$

$$= - E_c P_r \left[RM' + (PM' - g'R)\nu - P g' \nu^2\right] \tag{76}$$

Comparison of the coefficients of like terms on both sides gives

$$x^2 R_3'' + \left(5x - \frac{P_{e'}}{ac} x^2\right) R_3' + 4 R_3 = Z_4 (x) \tag{77}$$

$$x^2 R_2'' + \left(3x - \frac{P_{e'}\, x^2}{ac}\right) R_2' + R_2 = Z_3 (x) \tag{78}$$

and

$$R_1'' + \left(\frac{1}{x} - \frac{P_{e'}}{ac}\right) R_1' = Z_2 (x) \tag{79}$$

where

$$Z_2 (x) = \frac{2 M R_2'}{c} + \frac{1}{c}\left(M' + \frac{2M}{x}\right) R_2 - \left(\frac{2(1 + M^2)}{c^2}\right) R_3 - \frac{E_c P_r M'}{ac} R \tag{80}$$

$$Z_3 (x) = 4 a x^2 M\, R_3' + 2 a x^2 R_3 \left(M' + \frac{4M}{x}\right) - E_c P_r \left(x^2 P M' + c R\right) \tag{81}$$

$$Z_4 (x) = \frac{- E_c P_r}{a} P(x) \tag{82}$$

The solution of equation (79) is

$$R_1 (x) = \int \left\{ \frac{e^{\left(\frac{P_{e'}}{ac}\right) x}}{x} \int x\, e^{-\left(\frac{P_{e'}}{ac}\right) x} Z_2 (x)\, dx \right\} dx + C_5 \int \left\{ \frac{e^{\left(\frac{P_{e'}}{ac}\right) x}}{x} \right\} dx + C_6 \tag{83}$$

The solutions of equations (77) and (78) using Mathematica is

$$R_2 (x) = C_3 \left(-1 + \frac{1}{A1\, x}\right) + C_4 \text{ MeijerG}[\{\{\ \}, \{1\}\}, \{\{-1, -1\}, \{\}\}, -A1\, x]$$

$$+ \frac{1}{x} A1\, (-1 + A1\, x) \int e^{-A1 r} x \text{ MeijerG}[\{\{\ \}, \{1\}\}, \{\{-1, -1\}, \{\}\}, -A1\, x]\, Z_3 (x)\, dx$$

$$+ x \text{ MeijerG}[\{\{\ \}, \{1\}\}, \{\{-1, -1\}, \{\}\}, -A1\, x] \int A1 (1 - A1\, x)\, e^{-A1 x} Z_3 (x) dx\ \} \tag{84}$$

$$R_3 (x) = C_1 \left(\frac{1}{2} - \frac{2}{A1\, x} + \frac{1}{(A1\, x)^2}\right) + C_2 \text{ MeijerG}[\{\{\ \}, \{1\}\}, \{\{-2, -2\}, \{\}\}, -A1\, x]$$

$$+\frac{1}{x^{2}}\left\{A1^{2}\left(2-4\,A1\,x+A1^{2}\,x^{2}\right)\right.$$
$$\int e^{-A1\,x}\,x^{3}\,\text{MeijerG}[\{\{\quad\},\{1\}\},\ \{\{-2,-2\},\{\}\},\ -A1\,x]\ Z_{4}(x)\,dx$$
$$+x^{2}\,\text{MeijerG}[\{\{\quad\},\{1\}\},\ \{\{-2,-2\},\{\}\},\ -A1\,x]$$
$$\left.\int A1^{2}\left(-2+4\,A1\,x-A1^{2}\,x^{2}\right)x\,e^{-A1\,x}\,Z_{4}(x)\,dx\right\} \tag{85}$$

where $A1=\dfrac{P_{e'}}{a\,c}$ (86)

Therefore equations (83-85) in equation (75) provides temperature distribution for finite $P_{e'}$.

IV. CONCLUSION

Applying the successive transformation technique on non-dimensional equations for plane steady motion of incompressible fluids of variable viscosity in von-Mises coordinates a class of exact solutions are determined. In this class of flows the streamlines are characterized by equation $y=f(x)+g(x)\nu(\psi)$ with $\nu(\psi)=a\psi+b$ with constants $a\neq 0$ and b . Based on two velocity field shapes exact solutions are obtained. The first velocity field requires $g(x)=\dfrac{-1}{C_0x^2+C_1}$ and $f(x)$ satisfies a second order variable coefficients differential equation with a trivial solution $f(x)=0$. Therefore the streamlines for this case are $y=-\dfrac{a\psi+b}{C_0x^2+C_1}$. The second velocity field demands $g(x)=\dfrac{c}{x}$ and leaves the function $f(x)$ arbitrary. Therefore the streamlines for this case are $y=f(x)+\dfrac{c\,(a\psi+b)}{x}$. Both the velocity field cases create infinite set of streamlines, pressure, viscosity, generalized energy function, temperature distribution. The symbols $c\neq 0$, $C_0\neq 0$, C_1 , are constants.

The software Mathematica is used to determine the solution of some of the ordinary differential equations. The same software can easily draw the streamlines pattern to find the effect of various parameters on the streamlines and discuss the flow characteristic.

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