A Class of New Exact Solution of Equations for Motion of Variable Viscosity Fluid with Moderate Peclet Number through Von-Mises Coordinates

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Abstract

This communication announces a class of new exact solutions of the equations governing the steady plane motion of fluid with constant density, constant thermal conductivity but variable viscosity with moderate Peclet numbers through von.Mises coordinates. The class is characterized through an equation relating two functions of variable x and one function of stream function ψ. The successive transformation technique is applied to find unknowns functions in basic equations. This technique determines two temperature distribution formulas due to heat generation and corresponding viscosities. A large in number of exact solutions are shown for moderate Peclet number in von-Mises coordinates.

Keywords - *Variable viscosity fluids, Navier-Stokes equations with body force, Incompressible fluids, Moderate Peclet number, von-Mises coordinates*.

I. INTRODUCTION

The equations in non-dimensional form for theoretical study of steady fluid flow problem with constant density, constant thermal conductivity, constant specific heat but variable viscosity using following dimensionless parameters

$$x^* = \frac{x}{L_0} \qquad y^* = \frac{y}{L_0} \qquad u^* = \frac{u}{U_0} \qquad v^* = \frac{v}{U_0}$$

$$\mu^* = \frac{\mu}{\mu_0} \qquad p^* = \frac{p}{p_0} \qquad F_1^* = \frac{F_1}{F_0} \qquad F_2^* = \frac{F_2}{F_0}$$

after dropping the overhead "*" in tensor's notation are

$$\nabla . \mathbf{v} = \frac{\partial v_k}{\partial x_k} = 0 \tag{1}$$

$$\left(v_k \frac{\partial v_i}{\partial x_k} \right) = -\frac{\partial p}{\partial x_i} + \frac{1}{R_e} \frac{\partial}{\partial x_j} \left\{ \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right\} \tag{2}$$

$$\left(v_k \frac{\partial T}{\partial x_k} \right) = \frac{1}{R_e P_r} \frac{\partial}{\partial x_i} \left(\frac{\partial T}{\partial x_i} \right) + \frac{E_c}{R_e} \mu \frac{\partial v_i}{\partial x_j} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{3}$$

Where $\mathbf{v} = v_i(x_i)$ is the fluid velocity vector, $p = p(x_i)$ is pressure, and $\mu = \mu(x_i)$ is viscosity, $i, j, k \in \{1, 2, 3\}$. The dimensionless quantity R_e, P_r and E_c are known as the *Reynolds number*, the *Prandtl number* and the *Eckert number* respectively. The product of R_e and P_r is Peclet number denoted by $P_{e'}$. For the plane case $i, j, k \in \{1, 2\}$, $x_1 = x$, $x_2 = y$, $v_1 = u$ $v_2 = v$, reduces the equations (1-3) to following

$$u_x + v_y = 0 \tag{4}$$

$$u\, u_x + v\, u_y = -p_x + \frac{1}{\mathrm{R}_e} [(2\mu u_x)_x + \{\mu(u_y + v_x)\}_y] \tag{5}$$

$$u\,v_x + v\,v_y = -\,p_y + \frac{1}{\mathrm{R}_e}[(2\mu v_y)_y + \{\mu(u_y + v_x)\}_x] \tag{6}$$

$$u\,T_x + v\,T_y = \frac{1}{P_{e'}}(T_{xx} + T_{yy}) + \frac{\mathrm{E}_c}{\mathrm{R}_e}[2\mu(u_x^{\,2} + v_y^{\,2}) + \mu(u_y + v_x)^2] \tag{7}$$

The equation (4) implies that velocity components in terms of a continuous stream function $\psi(x, y)$ are

$$\mathbf{v} = (u, v) = (\psi_y, -\psi_x) \tag{8}$$

Like other mechanics, fluid dynamics also offers difficulties due to nonlinearity in the basic equations. To overcome these difficulties some methods/techniques are found in [1-7] and articles referred therein. References [6, 8-10] and reference therein are examples of solution techniques for a given Peclet number. This discourse uses method of partial differentiation and successive transformation technique in von-Mises coordinates for a class of exact solutions to flow equations (5-7). The technique first rewrites equation (5-7) in terms of the vorticity function Ω and the total energy function L, the function A and B defined as follow

$$\Omega = v_x - u_y \tag{9}$$

$$L = p + \frac{1}{2}(u^2 + v^2) - \frac{1}{\mathrm{R}_e}(2\mu u_x) \tag{10}$$

$$A = \mu(u_y + v_x) \qquad B = 4\mu\, u_x \tag{11}$$

Secondly, we take $\varphi(x, y) = \text{Const.}$ to be some arbitrary families of curves which generates a curvilinear net with the streamlines $\psi(x, y) = \text{Const.}$ so that in the physical plane the independent variables x, y can be replaced by (φ, ψ). Let

$$x = x(\varphi, \psi)\,, \qquad y = y(\varphi, \psi) \tag{12}$$

define a curvilinear net in the $(x, y)-$ plane with the squared element of arc length along any curve given by

$$dS^2 = E(\varphi, \psi)\, d\varphi^2 + 2F(\varphi, \psi)\, d\varphi\, d\psi + G(\varphi, \psi)\, d\psi^2 \tag{13}$$

wherein

$$E = x_\varphi^{\,2} + y_\varphi^{\,2}, \qquad F = x_\varphi x_\psi + y_\varphi y_\psi\,, G = x_\psi^{\,2} + y_\psi^{\,2} \tag{14}$$

such that the Jacobian of the transformation $J = \dfrac{\partial(x, y)}{\partial(\varphi, \psi)} \neq 0$ and finite. As this was the case in Martin [11] therefore these coordinates are referred here as Martin's coordinates system (φ, ψ). Let Martin's coordinates net (φ, ψ) possesses angle λ at a point $P(x, y)$ then

$$\tan(\lambda) = \frac{y_\varphi}{x_\varphi} \tag{15}$$

It is easy to transform the basic equations (5-7) in Martin's system [1].

II. BASIC EQUATIONS IN VON-MISES COORDINATES

In Martin's coordinates net (φ, ψ) the arbitrary coordinate lines $\varphi = \text{constant}$ can be taken along $x - axis$ so that x and ψ are independent variables, called von-Mises coordinates, instead of y and x [12]. Set

$$\varphi = x \tag{16}$$

in order to determine exact integrals for a class of flow problems in von-Mises coordinates and the streamline pattern for a given flow problem in the form

$$\frac{y - f(x)}{g(x)} = \text{Const.} \tag{17}$$

where $f(x)$ and $g(x) \neq 0$ are differentiable functions. For a given flow problem with $\frac{y - f(x)}{g(x)} = \text{Const.}$ as the family of streamlines, we have

$$y = f(x) + g(x)\nu(\psi) \tag{18}$$

with $\nu(\psi)$ is a differentiable functions.

Applying equation (16) and the streamlines (18) in equation (15) and equations (14) and using the trigonometric identities, we have

$$\cos\lambda = \frac{1}{\sqrt{E}} \tag{19}$$

$$E = 1 + (M + N\nu)^2 \tag{20}$$

$$F = J\sqrt{E-1} \tag{21}$$

$$G = J^2 \tag{22}$$

$$J = x\,g(x)\nu'(\psi) \tag{23}$$

where

$$M(x) = x\,f'(x) \qquad N(x) = x\,g'(x), \tag{24}$$

Utilizing (19-24), the basic flow equations (5-7), the vorticity function Ω, the function A and B are retransform to von-Mises coordinates system (x,ψ) as follow

$$-R_e\Omega = R_e L_\psi - JA_x + \sqrt{E-1}\,A_\psi + B_\psi \tag{25}$$

$$0 = -R_e L_x + \frac{A_\psi(2-E)}{J} + A_x\sqrt{E-1} - \frac{\sqrt{E-1}\,B_\psi}{J} \tag{26}$$

$$\begin{aligned} &J T_{xx} - 2\sqrt{E-1}\,T_{\nu x}\nu' + \frac{E}{J}T_{\nu\nu}(\nu')^2 + \left(J_x - \frac{E_\psi}{2\sqrt{E-1}} - P_{e'}\right)T_x \\ &+\left(\frac{E_\psi}{J} - \frac{E_x}{2\sqrt{E-1}} + \frac{EJ_\psi}{J^2} + \frac{E}{J}\left(\frac{\nu''}{\nu'}\right)\right)T_\nu\nu' = -\frac{J\,E_c\,P_r}{4\mu}\left(B^2 + 4A^2\right) \end{aligned} \tag{27}$$

$$\begin{aligned} \Omega = &\left[\frac{(x f'' + f')}{x g} - \frac{2 f' g'}{g^2}\right]\left(\frac{1}{\nu'}\right) + \left[\frac{(x g'' + g')}{x g} - \frac{2(g')^2}{g^2}\right]\left(\frac{\nu}{\nu'}\right) \\ &+\left(\frac{1}{x^2 g^2} + \frac{f'^2}{g^2}\right)\left(\frac{\nu''}{(\nu')^3}\right) + \left(\frac{2 f' g'}{g^2}\right)\left(\frac{\nu\nu''}{(\nu')^3}\right) + \frac{g'^2}{g^2}\left(\frac{\nu^2\nu''}{(\nu')^3}\right) \end{aligned} \tag{28}$$

$$\begin{aligned} A = &\frac{\mu}{x^2 g^2 \nu'}\,[x g\,[(x f'' + f') + (x g'' + g')\nu)] \\ &-\{1 - x^2(f' + g'\nu)^2\}\left(\frac{\nu''}{\nu'^2}\right) - 2x(f' + g'\nu)(x g' + g)] \end{aligned} \tag{29}$$

$$B = \frac{4\mu}{x^2 g^2 \nu'}\left[-(x g' + g) + x(f' + g'\nu)\left(\frac{\nu''}{\nu'^2}\right)\right] \tag{30}$$

The momentum equations (25-26) on eliminating L applying $L_{x\psi} = L_{\psi x}$, provides

$$JA_{xx} - 2\sqrt{E-1}\,A_{x\psi} - \frac{(2-E)}{J}A_{\psi\psi} + A_x\left(J_x - \frac{E_\psi}{2\sqrt{E-1}}\right)$$

$$+ A_{\psi}\left(-\frac{E_x}{2\sqrt{E-1}}+\frac{J_{\psi}(2-E)}{J^2}+\frac{E_{\psi}}{J}\right)-\left\{B_x-\frac{\sqrt{E-1}\,B_{\psi}}{J}\right\}_{\psi}=R_e\,\Omega_x \tag{31}$$

III. EXACT SOLUTION IN VON-MISES COORDINATES

The compatibility equation (31) is to lead to the solution of (25-26) and as this discourse is considering $v''(\psi)\neq 0$ and $g'(x)\neq 0$ therefore it is reasonable to set

$$v(\psi)=e^{\psi}, \tag{32}$$

$$g=\frac{c}{x}, \tag{33}$$

in equations (28-30) leads to

$$f(x)=\ln x+b \tag{34}$$

where b and c are constant. Equations (32-34) simplifies the functions Ω, A and B as follow

$$\Omega=\frac{2}{c^2e^{2\psi}}, \tag{35}$$

$$A=-\frac{2\mu}{c\,x\,e^{\psi}}\left(1-\frac{ce^{\psi}}{x}\right) \tag{36}$$

and

$$B=\frac{4\mu}{c^2e^{\psi}}\left(1-\frac{c\,e^{\psi}}{x}\right)\left(\frac{1}{e^{\psi}}\right) \tag{37}$$

It is easy to eliminate μ from equations (36-37) and find

$$B=\frac{-2x}{c\,e^{\psi}}A \tag{38}$$

Substituting equation (32-34) and equation (38), the equation (31) reduces to

$$c\,v\,A_{xx}+\left(-2v+\frac{2cv^2}{x}+\frac{2x}{c}\right)A_{xv}+\left(-\frac{2x}{c^2}+\frac{2v}{c}-\frac{2v^2}{x}+\frac{cv^3}{x^2}\right)A_{vv}$$
$$+\left(\frac{cv}{x}-\frac{2x}{cv}\right)A_x+\left(-\frac{2v}{x}+\frac{cv^2}{x^2}+\frac{4x}{c^2v}\right)A_v-\frac{4x}{c^2v^2}A=0 \tag{39}$$

Equation (39) suggest to seek numbers (n,m) and p such that

$$A(x,\psi)=p\,x^n v^m \tag{40}$$

Equation (39) on substituting equation (40) provides

$$p\,x^n v^m\ [\frac{v}{x^2}\{cn(n-1)+2cn\,m+cm(m-1)+nc+cm\}$$
$$+\frac{1}{v}\left\{\frac{2nm}{c}+\frac{2m(m-1)}{c}-\frac{2n}{c}\right\}+\frac{1}{x}\{-2nm-2m(m-1)-2m\}$$
$$\frac{x}{v^2}\left\{-\frac{2m(m-1)}{c^2}+\frac{4m}{c^2}-\frac{4}{c^2}\right\}]\qquad=0 \tag{41}$$

Equation (41) implies

$$n(n-1)+2nm+m(m-1)+n+m=0 \tag{42}$$

$$nm+m(m-1)-n=0 \tag{43}$$

$$nm+m(m-1)+m=0 \tag{44}$$

$$m\,(m-1) - 2m + 2 \quad =0 \tag{45}$$

The system of algebraic equations (42-45) implies

$$m = 1,\ 2 \qquad \text{and} \qquad n = -m$$

For the numbers $(m, n) = (1, -1)$, equation (36), equation (40) and equations (25-26) provides

$$\mu = \left(\frac{-pc}{2}\right)\left(\frac{x\nu^2}{(x - c\nu)}\right) \tag{46}$$

$$A = p\left(\frac{\nu}{x}\right) \tag{47}$$

and

$$R_e\, L = \left(\frac{R_e}{c^2 \nu^2}\right) - p\left(\frac{\nu}{x}\right) + p_1 \tag{48}$$

where p_1 is constant. The energy equation (27) for this case is

$$\begin{aligned}(c\nu)^2\, T_{xx} - 2\left(1 - \frac{c\nu}{x}\right) c\nu^2\, T_{\nu x} + \nu^2\left\{2 - \frac{2c\nu}{x} + \frac{c^2\nu^2}{x^2}\right\} T_{\nu\nu} \\ + \left(\frac{c^2\nu^2}{x} - c\nu\, P_{e'}\right) T_x + \left(-\frac{2c\nu^2}{x} + \frac{c^2\nu^3}{x^2}\right) T_\nu \\ = \frac{2\,D\,E_c\,P_r}{c}\left[1 - \left(\frac{c\nu}{x}\right) + \left(\frac{c\nu}{x}\right)^2 - \left(\frac{c\nu}{x}\right)^3\right]\end{aligned} \tag{49}$$

Equation (49) suggests T of the form

$$T = a \ln x + b \ln \nu + T_0 + T_1\left(\frac{\nu}{x}\right) + T_2\left(\frac{\nu}{x}\right)^2 \tag{50}$$

Equation (49) on inserting equation (50) provides

$$\begin{aligned}&\left(\frac{\nu}{x}\right)^4 \left\{c^2(6T_2) + 2c^2(-4T_2) + c^2(2T_2) + c^2(-2T_2) + c^2(2T_2)\right\} \\ &+\left(\frac{\nu}{x}\right)^3 \left\{\begin{aligned}&c^2(2T_1) + 2c^2(-T_1) - 2c(-4T_2) - 2c(2T_2) \\ &\qquad + c^2(-T_1) - c\,P_{e'}(-2T_2) + c^2(T_1) - 2c(2T_2)\end{aligned}\right\} \\ &+\left(\frac{\nu}{x}\right)^2 \left\{\begin{aligned}&c^2(-a) - 2c(-T_1) + c^2(-b) + 2(2T_2) + c^2(a) - c\,P_{e'}(-T_1) \\ &\qquad + c^2(b) - 2c(T_1)\end{aligned}\right\} \\ &+\left(\frac{\nu}{x}\right)\left\{-2c(-b) - c\,P_{e'}(a) - 2c(b)\right\} + \left\{2(-b)\right\} \\ &= \frac{2E_c P_r D}{c}\left\{1 - \left(\frac{c\nu}{x}\right) + \left(\frac{c\nu}{x}\right)^2 - \left(\frac{c\nu}{x}\right)^3\right\}\end{aligned} \tag{51}$$

Comparing coefficients of like terms yields

$$a = \frac{2E_c\,p}{c\,P_r}, \quad b = -\frac{E_c P_r\,p}{c},\ T_1 = \frac{2\,p\,E_c P_r\,(P_{e'} + 2)}{(P_{e'})^2},\ T_2 = \frac{-c\,p\,E_c}{R_e} \tag{52}$$

Therefore, from equation (50), we get

$$T = \frac{2E_c\,p}{cR_r}\ln x - \frac{p\,E_c P_r}{c}\ln \nu + T_0 + \frac{2\,p\,E_c P_r\,(P_{e'}+2)}{(P_{e'})^2}\left(\frac{\nu}{x}\right) - \frac{c\,p\,E_c}{R_e}\left(\frac{\nu}{x}\right)^2 \tag{53}$$

Thus temperature form equation (53), viscosity from (46), pressure from (10) using (48) and velocity from equation (8) for moderate Peclet number is obtained.

For the numbers $(m, n) = (1, -1)$ the function A, μ and L are

$$A = p\left(\frac{\nu}{x}\right)^2 \tag{54}$$

$$\mu = \left(\frac{-p\,c}{2}\right)\left(\frac{\nu^3}{(x - c\,\nu)}\right) \tag{55}$$

$$R_e\,L = \left(\frac{R_e}{c^2\,\nu^2}\right) - \left(\frac{p\,\nu^2}{x^2}\right) + \left(\frac{2\,p\,\nu}{c\,x}\right) + \left(\frac{2\,p}{c^2}\right)\ln x + p_2 \tag{56}$$

where p_2 is constant.

The energy equation (27) becomes

$$\begin{aligned}
(c\,\nu)^2\,T_{xx} - 2\left(1 - \frac{c\,\nu}{x}\right)c\,\nu^2\,T_{\nu x} + \nu^2\left\{2 - \frac{2c\,\nu}{x} + \frac{c^2\,\nu^2}{x^2}\right\}T_{\nu\nu} \\
+ \left(\frac{c^2\,\nu^2}{x} - c\,\nu\,P_{e'}\right)T_x + \left(-\frac{2c\,\nu^2}{x} + \frac{c^2\,\nu^3}{x^2}\right)T_\nu \\
= \frac{2\,D\,E_c\,P_r}{c}\left[\left(\frac{\nu}{x}\right) - c\left(\frac{\nu}{x}\right)^2 + c^2\left(\frac{\nu}{x}\right)^3 - c^3\left(\frac{\nu}{x}\right)^4\right]
\end{aligned} \tag{57}$$

which on substituting

$$T = a\ln x + T_0 + T_1\left(\frac{\nu}{x}\right) + T_2\left(\frac{\nu}{x}\right)^2 + T_3\left(\frac{\nu}{x}\right)^3 \tag{58}$$

yields

$$a = -\frac{2\,p\,E_c}{c^2\,R_e}, \qquad T_1 = \frac{-p\,E_c}{c\,R_e}\left(1 + \frac{4}{P_{e'}} - \frac{8}{(P_{e'})^2}\right),$$

$$T_2 = \frac{p\,E_c}{R_e}\left(1 + \frac{4}{P_{e'}}\right), \qquad \text{and} \qquad T_3 = \frac{-2\,p\,E_c}{3\,R_e} \tag{59}$$

Thus equation (58) gives

$$\begin{aligned}
T = \left(-\frac{2\,p\,E_c}{c^2\,R_e}\right)\ln x + T_0 + \left(\frac{-p\,E_c}{c\,R_e}\right)\left(1 + \frac{4}{P_{e'}} - \frac{8}{(P_{e'})^2}\right)\left(\frac{\nu}{x}\right) \\
+ \left(\frac{p\,E_c}{R_e}\right)\left(1 + \frac{4}{P_{e'}}\right)\left(\frac{\nu}{x}\right)^2 + \left(\frac{-2\,p\,E_c}{3\,R_e}\right)\left(\frac{\nu}{x}\right)^3
\end{aligned} \tag{60}$$

Thus temperature form equation (60), viscosity from (55), pressure from (10) using (56) and velocity from equation (8) for moderate Peclet number is obtained.

IV. CONCLUSION

A class of new exact solutions of the equations governing the steady plane motion of fluid of constant density, constant thermal conductivity but variable viscosity in von-Mises coordinates for moderate Peclet numbers is obtained. The streamline pattern for class of flows under consideration is found to be $y = b + \ln x + \dfrac{c\ e^{\psi}}{x}$. Where stream function is ψ. Two temperature distribution formulas due to heat generation are obtained for corresponding viscosities for moderate Peclet number. This discourse shows streamlines pattern, velocity components, viscosity function, and temperature distribution to the flow problem for all moderate Peclet number.

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