Characterization of Some Graphs using Graph Equations

Beena S
Associate Professor, Department of Mathematics
NSS College, Nilamel, Kollam, Kerala, India

Abstract
The line graph of a graph G denoted by $L(G)$ is defined as the graph whose vertices are the edges of G and where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are incident to a common vertex. A connected graph G is said to be unicyclic if it contains a unique cycle. In this work, some unicyclic graphs, paths and generalized 3 -stars are characterized using graph equations involving line graphs.

Keywords - *Line graphs, Graph equations.*

I. INTRODUCTION

All graphs considered in this paper are finite, undirected, without loop or multiple edges. Let p, q and $d_1, d_2, ..., d_p$ respectively denote the order, size, and degree sequence of a given graph G.

Definition 1. The line graph of a graph G denoted by $L(G)$ is defined as the graph whose vertices are the edges of G and where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are incident to a common vertex.

Theorem 2 [2]. If G is a (p, q) graph and degree sequence $d_1, d_2, ..., d_p$, then $L(G)$ has q vertices and q_L edges where $q_L = -q + \frac{1}{2}\sum_{i=1}^{p} d_i^2$.

Definition 3. The n – power G^n of a graph G has the same vertex set as G and two vertices u and v are adjacent in G^n if and only if $d(u, v) \le n$ in G.

Theorem 4[1]. A connected graph G with order $p \ge 2$ and degree sequence $d_1, d_2, ..., d_p$ is a path if and only if $\sum_{i=1}^{p} d_i^2 = 4p - 6$.

Definition 5. A connected graph with one vertex of degree 3, three vertices of degree 1 and $p-4$ vertices of degree 2 is called a generalized 3-star of order p.

Theorem 6[1]. A connected graph G with order $p \ge 2$ and degree sequence $d_1, d_2, ..., d_p$ is a generalized 3-star if and only if $\sum_{i=1}^{p} d_i^2 = 4p - 4$.

II. MAIN RESULTS

Theorem 7. There does not exist a connected (p, q) graph G satisfying the graph equation $L(G) \cong G - e$.

Proof: Let G be a connected (p, q) graph, then $L(G)$ is a (q, q_L) graph where $q_L = -q + \frac{1}{2}\sum_{i=1}^{p} d_i^2$. Since $L(G) \cong G - e$, we have $q = p$ and $q_L = q - 1$. Therefore, G is a unicyclic graph and $q - 1 = -q + \frac{1}{2}\sum_{i=1}^{p} d_i^2$, which implies $\sum_{i=1}^{p} d_i^2 = 4p - 2$. That is, G is a generalized double 3 – star by theorem 6, which is a contradiction to the fact that G is a unicyclic graph. Hence there exists no connected graph G satisfying the equation $L(G) \cong G - e$.

Theorem 8. Let G be a connected (p,q) graph. Then $L(G) \cong G - \{v\}$ for some $v \in L(G)$ if and only if G is a path.

Proof: Let G be a connected (p,q) graph. Let v be a vertex of G. Suppose that $L(G) \cong G - \{v\}$. Then $L(G)$ is a $(p-1, q_L)$ graph. Thus G is a connected $(p, p-1)$ graph. Hence G is a tree and v must be a pendant vertex of G. Further, $q - 1 = -q + \frac{1}{2}\sum_{i=1}^{p} d_i^2$. This implies, $\sum_{i=1}^{p} d_i^2 = 4q - 2 = 4p - 6$. Hence G is a path, by theorem 4. The converse is trivial.

Theorem 9. Let G be a connected (p,q) graph. Then $L(G^2) \cong (L(G))^2$ if and only if G is K_3 or K_2.

Proof: Let G be a connected (p,q) graph, then $L(G)$ is a (q, q_L) graph where $q_L = -q + \frac{1}{2}\sum_{i=1}^{p} d_i^2$ and $(L(G))^2$ has q vertices. Suppose that $L(G^2) \cong (L(G))^2$. Therefore, $L(G^2)$ must have q vertices. Therefore, G^2 is also (p,q) graph. This implies, $G = G^2$. Therefore $G = K_p$ and $L(G^2) \cong (L(G))^2$ implies $L(G) \cong (L(G))^2$. Therefore, $L(G)$ is a complete graph. But $L(G)$ is a $\left(\frac{p(p-1)}{2}, q_L\right)$ graph, where $q_L = -\frac{p(p-1)}{2} + \frac{1}{2}p(p-1)^2 = \frac{p}{2}(p-1)(p-2)$. Also $q_L = \frac{1}{2}\frac{p(p-1)}{2}\left(\frac{p(p-1)}{2} - 1\right)$. Therefore, $\frac{p}{2}(p-1)(p-2) = \frac{p(p-1)}{4}\left(\frac{p(p-1)-2}{2}\right)$. That is, $p - 2 = \frac{p(p-1)-2}{2}$ or $p^2 - 5p + 6 = 0$.

That is, $p = 2$ or 3. Therefore, $G = K_3$ or K_2.

Conversely, If $G = K_3$ or K_2, then $L(G^2) \cong (L(G))^2$.

Theorem 10. Let G be a connected (p,q) graph which is not a cycle, a path or a generalized $3-$ star. Let $q_n = q(L^n(G))$ where $L^0(G) \cong G$. Then $\{q_n\}_{n=0,1,2,\ldots}$ is a strictly increasing sequence.

Proof: Let G be a connected (p,q) graph, then $L(G)$ is a (q, q_1) graph. Since G is connected ($L(G)$ also), $q_1 \geq q - 1$. If $q_1 = q - 1$, $L(G)$ must be a tree. Which contradicts the fact that G is not a path. If $q_1 = q$, Then $L(G)$ is a unicyclic graph. Clearly $q \geq p - 1$. If $q > p$, then $L(G)$ will have more than one cycle, which is a contradiction. Therefore $q = p - 1$ or p. Also $q_1 = q$ implies $\sum_{i=1}^{p} d_i^2 = 4q = 4p - 4$ or $4p$.

$\sum_{i=1}^{p} d_i^2 = 4p - 4$ implies, G is a generalized 3-star and $\sum_{i=1}^{p} d_i^2 = 4p$, implies G is a cycle. That is $q_1 = q$ implies G is a cycle or a generalized 3-star, a contradiction to our assumption. Hence, $q_1 > q$.

Suppose we have proved that for any graph not in this class, then $q < q_1 < \ldots < q_n$. Now, $L(G)$ can not be a generalized 3-star, which has $K_{1,3}$ as an induced subgraph. Suppose $L(G) \cong C_n, n \geq 3$. First suppose $n \geq 4$, then $L(G) \cong C_n \cong L(C_n)$ which implies $G \cong C_n$, which is a contradiction. Therefore, $L(G) \cong C_3$, implies $G \cong C_3$ or $K_{1,3}$, again a contradiction. Therefore, if G does not belong to the class of graphs so does $L(G)$. Therefore by induction, $q(L^{n+1}(L(G))) < q(L^n(L(G)))$. This implies, $q(L^n(G)) < q(L^{n+1}(G))$. Hence the proof.

Corollary 11. Let G be a connected (p,q) graph, then $L^m(G) \cong L^n(G)$ for $m > n$ if and only if G is a cycle or $K_{1,3}$.

Proof: By theorem 10, G must be either a cycle, a path or a generalized 3-star. Clearly G is not a path. Therefore, G is either a cycle or a generalized 3-star. If G is a generalized 3-star with $p > 4$, then $L(G)$ is a unicyclic graph other than cycle. Therefore, $q(L^n(L(G))) = q(L^{n+1}(G))$ is an increasing sequence for $n = 1,2,\ldots$

by theorem 10.In this case $L^m(G) \tilde{\not\cong} L^n(G)$ for $m > n$. Hence G must be a cycle or a generalized 3-star with $p = 4$. That is $K_{1,3}$.

The converse is trivial.

Corollary 12. Let G be a tree on p vertices. Then G is a generalized 3-star if and only if $q(G) = q(L(G))$.

Proof: By theorem 6, G is a generalized 3-star if and only if $\sum_{i=1}^{p} d_i^2 = 4p - 4$. Also we have,

$q(L(G)) = -q + \frac{1}{2}\sum_{i=1}^{p} d_i^2$. That is, $\sum_{i=1}^{p} d_i^2 = 2(q + q(L(G)))$. Therefore, G is a generalized 3-star if and only if

$q(L(G)) = 2p - 2 - q = 2p - 2 - (p-1) = p - 1 = q = q(G)$

Corollary 13. Let G be a connected (p, q) graph, then $L^m(G^n) \cong L^r(G^n)$ where $m > r, n > 1$ if and only if G is C_3 or $K_{1,3}$.

Proof: Let $L^m(G^n) \cong L^r(G^n)$ where $m > r, n > 1$. Then by corollary 5, G^n is a cycle or $K_{1,3}$ .Therefore, G is C_3 or $K_{1,3}$.

Theorem 14. Let G be a connected (p, q) graph, then $L(G^n) \cong L(G)$ for some $n > 1$ if and only if G is the complete graph K_p.

Proof: Let G be a connected (p, q) graph. Then G^n is a $(p, q(G^n))$ graph. Suppose that $L(G^n) \cong L(G)$.

This implies that $q(G^n) = q(G) \forall n > 1$ and therefore, $G = K_p$. The converse is trivial.

III. CONCLUSION

There does not exist a connected graph G satisfying the graph equation $L(G) \cong G - e$. A connected (p, q) graph G is a path if and only if $L(G) \cong G - \{v\}$ for some $v \in L(G)$, is K_3 or K_2 if and only if $L(G^2) \cong (L(G))^2$, is a cycle or $K_{1,3}$ if and only if $L^m(G) \cong L^n(G)$ for $m > n$, is C_3 or $K_{1,3}$ if and only if $L^m(G^n) \cong L^r(G^n)$ where $m > r, n > 1$, is the complete graph K_p if and only if $L(G^n) \cong L(G)$ for some $n > 1$ and is a generalized 3-star if and only if $q(G) = q(L(G))$.

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