Quasilinear Delay Differential Equations with Nonlocal Conditions via Measures of Non Compactness

Francis Paul Samuel
Department of Mathematics, Chemistry and Physics,
University of Eastern Africa, Baraton,
Eldoret 2500 - 30100 Kenya

Abstract
In this paper, we study the existence of mild solutions for quasilinear delay differential equations with nonlocalCauchy problem in Banach spaces. The results are established by using Hausdorff's measure of noncompactness.

Keywords - *Mild solution, nonlocal conditions,noncompact measures, semigroup theory*

I. INTRODUCTION

The purpose of this paper is to study the existence of mild solutions of quasilinear delay integrodifferential equations with nonlocal condition of the form

$$u'(t) + A(t,u)u(t) = f\left(t, u(\alpha(t))\right), \quad t\epsilon[0,T] \tag{1}$$

$$u(0) + g(u) = u_0, \tag{2}$$

Where $A : [0,T] \times X \rightarrow X$ are continuous functions inBanach space X,
$u_0 \epsilon X, f: [0,T] \times X \rightarrow X, \; g: C([0,T]; X) \rightarrow X$ and α are given function.
Here $\Delta = \{t, s, \; 0 \le s \le t \le T\}$.

The notion of "nonlocal condition" has been introduced to extend the study of the classical initial value problems; see e.g. [4, 5, 10, 15]. It is more precise for describing nature phenomena than the classical condition since more information is taken into account, thereby decreasing the negative effects incurred by a possibly erroneous single measurement taken at the initial time. The study of abstract nonlocal initial value problems was initiated by Byszewski, we refer to some of the papers below. Byszewski[4], Byszewski and Lakshmikantham [5] give the existence and uniqueness of mild solutions and classical solutions when f and g satisfy the Lipschitz -type conditions.

The notion of a measure of noncompactness turns out to be a very important and useful tool in many branches of mathematical analysis. The notion of a measure of weak compactness was introduced by De Blasi [7] and was subsequently used in numerous branches of functional analysis and the theory of differential and integral equations. El-Sayed [9] proves the existence theorem of monotonic solutions for a nonlinear functional integral equation of convolution type by Hasusdorff measure of noncompactness.

Fan et al [10] discussed semilinear differential equations with nonlocal condition using measure of noncompactness. Paul Samuel et al [12, 13] give the existence results for quasilinear delay differential and integrodifferential equations with nonlocal conditions in Banach spaces. The problem of existence of solutions of quasilinear evolution equations in Banach space has been studied by several authors [6, 8, 13, 14].Pazy [14] considered the following quasilinear equation with local condition of the form

$$u'(t) + A(t,u)u(t) = 0, \qquad 0 < t < T$$
$$u(0) = u_0,$$

and discussed the mild and classical solutions by using the fixed point argument. The results obtained in this paper generalize the results of [7, 8, 12].

II. PRELIMINARIES

Let X be a Banach space with norm $||\cdot||$. Let $C([0,T];X)$ be the space of X-valued continuous functions on $[0,T]$ with the norm $||u|| = \sup\{||u(t)||, t \epsilon [0,T]\}$for u$\epsilon$ $C([0,T];X)$, and denoted $L(0,T;X)$ by thespace of X-valued Bochner integrable functions on $[0,T]$ with the norm $||u||_L = \int_0^T ||u(t)|| dt$. The Hausdorff's measure of noncompactness χY is defined by χ(B) = $\inf\{r > 0, B$ can be covered by finite number of balls with radii $r\}$ for bounded set B in a Banach space Y.

Lemma 2.1 [4]. Let Y be a real Banach space and$B, E \subseteq Y$ be bounded, with the following properties:
(i) B is precompact if and only if $\chi_X(B) = 0$.

(ii) $\chi_Y(B)$ = $\chi_Y(\bar{B})$ = $\chi_Y(conv\ B)$, where B and y $conv\ B$ means the closure and convex hull of B respectively.

(iii) χ_Y (B) ≤ χ_Y (E),where $B \subseteq E$.

(iv) χ_Y (B + E)) ≤ χ_Y (B) + χ_Y (E),
where$B + E = \{x + y : x \epsilon B, y \epsilon E\}$

(v) $\chi_Y(B \cup E) \leq \max\{\chi_Y(B), \chi_Y(E)\}$.

(vi)$\chi_Y(\lambda B) \leq |\lambda| \chi_Y(B) for\ any\ \lambda \in R$

(vii) If the map $F: D(F) \subseteq Y \to Z$ is Lipchitz continuous with constant k the $\chi_Y(FB) \leq k\chi_Y(B)$ for any bounded subset $B \subseteq D(F)$, where Z is Banach space.

(viii) $\chi_Y(B) = \inf\{d_Y(B,E); E \subseteq Y$is finite valued, where $d_Y(B,E)$ means the nonsymmetric (or symmetric) Hausdorff distance between B and E in Y.

(viii) If $\{W_n\}_{n=1}^{+\infty}$ is a decreasing sequence of bounded closed nonempty subset of Y and $\lim_{n\to\infty} \chi_Y(W_n) = 0$, then $\bigcap_{n=1}^{+\infty} W_n$ is nonempty and compact in Y.

Lemma 2.2 (Darbo-Sadovskii [4]). If W$\subseteq$ Y is bounded closed and convex, the continuous map $F : W \to W$is a χ_Y contraction, then the map F has atleast one fixed point in W.In this we denote by χ the Hausdorff's measure of noncompactness of X and denote _c by the Hausdorff'smeasure of noncompactness of $C([a,T];X)$.
To discuss the existence, we need the following Lemmas in this paper.

Lemma 2.3. [4]. If $W \subseteq C([0,T];X)$ is bounded. Then χ (W(t)) ≤ χ_c(W) for all $t \epsilon [0,T]$, where $W(t) = \{u(t);\ u \epsilon W\} \subseteq X$. Furthermore if W isequicontinuous on $[a,T]$, then $\chi\ (W(t))$ is continuous on$[a,T]$ and $\chi_c(W) = sup\{\ \chi\ (W\ (t)), t \epsilon [a,T]\}$.

Lemma 2.4. If $\{u_n\}_{n=1}^{\infty} \subset L^1(a,T;X)$ is uniformlyintegrable, then the function $\chi(\ \{u_n\}_{n=1}^{\infty})$ is measurable and
$\chi(\{\int_0^t u_n(s)ds\}) \leq 2\int_0^t \chi\ \{u_n\}_{n=1}^{\infty} ds\ )$ (3)

Lemma 2.5. [4].If $W \subseteq C([0,T];X)$ is bounded and equicontinuous, then then $\chi\ (W(t))$ is continuous and

$$\chi\left(\{\int_0^t W(s)ds\}\right) \leq \int_0^t \chi\ \mathrm{W(s)}ds \qquad (4)$$

for all $t \epsilon [0,T]$, where $\int_0^t$ W(s)ds)= $\{\int_0^t$ u(s)$ds; u \epsilon W\}$.

The C_0 - semigroup $U_u(t,s)$ is said to be equicontinuous for $t > 0$ for all bounded set B in X.The following lemma is obvious.

Lemma 2.6.If the evolution family$\{U_u(t,s)\}_{0\leq s\leq T}$isequicontinuous and $\eta \in L(0,T;R^+)$, then the set $\{\int_0^t U_u(t,s)u(s)ds\}, ||u(s)|| \leq \eta(s)$ for a.e s $\in [0,T]$ is equicontinuous for t $\in [0,T]$ .From [8], we know that for any fixed $\in C([0,T];X)$, there exist a unique continuous function $U_u: [0,T]$ x$[0,T] \to B(X)$ defined on $[0,T]$x$[0,T]$ such that

$U_u(t,s) = \int_s^t A_u U_u(\mathrm{w},s)dw$, (5)

where$B(X)$ denote the Banach space of bounded linear operators from X to X with the norm $||F|| = \sup\{||Fu||: ||u|| = 1\}$, and I stands for the identity operator on X, $A_u(t) = A(t,u(t))$.We have $U_u(t,t) = I$, $U_u(t,s)U_u(s,r) = U_u(t,r)$, where $(t,s,r) \in [0,T]$ x $[0,T]$x$[0,T]$,

$\frac{\partial U_u(t,s)}{\partial t} = A_u(t)U_u(t,s)$for almost all $\in [0,T]$.

For a mild solution of (1) - (2) we mean a function$u \in C([0,T];X)$ and $u_0 \in X$ satisfying the integral equation

$$u(t) = U_u(t,0)\,u_0 - U_u(t,0)g(u) + \int_0^t U_u(t,s)[f(s,u(\alpha(s))]ds, \quad (6)$$

III. EXISTENCE OF RESULT

In this section, we present and prove the existence results when g is compact and f satisfy the conditions with respect to Hasudorff's measure of noncompactness and its applications in differential equations in Banach spaces. We give some existence results of the nonlocal problem (1) - (2). We assume the following assumptions:

(H1) The evolution family $\{U_u(t,s)\}_{0\leq s\leq t\leq T}$ generated by $A(t,u)$ is equicontinuous, and $||U_u(t,s)|| \leq M_0$for almost$t,s \in [0,T]$.

(H2)(a) $f : [0,T]\times X \to X$ satisfies the Caratheodorytypeconditions and there exist $m_1 \in L[0,T]$ and $b_1 \geq 0$ such that

$$|f(t,u)| \leq m_1(t)b_1||u||, t \text{ a.e in } [0,T], u \in R^+$$

(b) There exist $\eta \in L(0,T;R^+), \zeta \in L(0,T;R^+)$, such that $\zeta(f(t,D) \leq \eta$ (t) $\chi(D)$ for a.e $t \in [0,T]$, and for any bounded subset $D \subset C([0,T],X)$, here we let $\eta(\text{t}) \leq K_0$.

(H3) (a) $g : C([0,T];\ X) \to X$ is continuous and compact.

(b) There exist $N > 0$ such that $||g(u)|| \leq N_0$ for all $u \in C([0,T];X)$.

(H4) $\alpha: [0,T] \to [0,T]$ is non-decreasing and there exist positive constant δ_1 such that $\alpha'(t) \geq \delta_1$for $t \in [0,T]$.

Theorem: 3.1. Assume that the conditions $(H1) - (H2)$are satisfied, then the nonlocal initial value problem. Let $u_0 \in Y$ and the family $A(t,u)$ of linear operators for $t \in I\ [0,T]$ and $(1) - (2)$ has at least one mild solution.

Proof.

Let $\Omega(t)$ be a solution of the scalar equation

$\Omega(t) = M_0[N_0 + M_0[m_1(t)b_1/\delta_1]]$for $t \in [0,T]$. (7)

Consider the map $F : C([0,T];X) \to C([0,T];X)$defined by

$(Fu)(t) = U_u(t,0)g(u) + \int_0^t U_u(t,s)\left[f\left(s,u(\alpha(s))\right)\right]ds$ (8)

for all $u \in C([0,T;X)$. We can show that F is continuousby the usual techniques.

Let us take

$$W_0 = \{u \in C([0,T];X), |\,|u(t)|\,| \leq \Omega(t), \quad \forall \quad t \in [0,T]\}.$$

Then $W_0 \subseteq C([0,T];X)$ is bounded and convex. We define $W_1 = \overline{conv}\, K(W_0)$, where $conv$ means the closure of the $convex$ hull in $C([0,T];X)$. As $U_u(t,s)$is equicontinuous, g is compact and $W_0 \subseteq C([0,T];X)$ is bounded, due to Lemma 2.7 and the assumption (H2) (b), $W_1 \subseteq C([0,T];X)$ is bounded closed convex nonempty and equicontinuous on $[0,T]$.

For any $u \in F(W_0)$, we know

$$||u(t)|| \leq M_0N_0 + M_0[\int_0^t \left\|f\left(s, u(\alpha(s))\right)\right\| ds$$

$$\leq M_0N_0 + M_0[\int_0^t m_1(s)b_1||u(\alpha(s)||ds]$$

$$\leq M_0N_0 + M_0[m_1(t)b_1 \int_0^t ||u(\alpha(s)|| \frac{\alpha^{'}(s)}{\delta_1 ds}]$$

$$\leq M_0N_0 + M_0[m_1(t)b_1/\delta_1 \int_0^t ||u((s)||ds]$$

$$\leq M_0[N_0 + M_0[m_1(t)b_1/\delta_1]]$$

= Ω(t) , for $t, s, \tau \in [0, T]$.

It follows that $W_1 \subset W_0$. We define $W_{n+1}=\overline{conv}\, F(W_n)$, for $n = 1, 2, 3, \cdots$ From above we know that $\{W_n\}_{n=1}^{\infty}$ is a decreasing sequence of bounded, closed, convex, equi-continuous on $[0, T]$ and nonempty subsets in $C([0, T]; X)$.

Now for $n \geq 1$ and $t \in [0, T], W_n(t)$ and $F(W_n(t))$ are bounded subsets of X, hence, for any $\in > 0$, there is a sequence $\{u_k\}_{k=1}^{\infty} \subseteq W_n$ such that [3], $\chi(W_{n+1})) = \chi(FW_n(t))$

$$\leq \chi(\int_0^t U_u(t, s)[f(s, \{u_k\alpha(s))\}_{k=1}^{\infty})])ds$$

$$\leq 2M_0 \int_0^t [\chi\,(f(s, \{u_k\alpha(s))\}_{k=1}^{\infty})ds]$$

$$\leq 2M_0K_0 \int_0^t \chi(\{u_k\alpha(s)\}_{k=1}^{\infty})ds$$

$$\leq 2M_0K_0/\delta_1 \int_0^t \chi(\{u_k(s)\}_{k=1}^{\infty})ds] + \epsilon$$

$$\leq 2M_0K_0\left[\frac{1}{\delta_1}\right]\int_0^t \chi(\{u_k(s)\}_{k=1}^{\infty})ds + \epsilon$$

$$\leq 2M_0K_0\left[\frac{1}{\delta_1}\right]\int_0^t \chi\left(W_n(s)\right)ds + \epsilon$$

Since ϵ > 0 is arbitrary, it follows that from the above inequality that

$$\chi(W_{n+1}(t))) \leq 2M_0K_0\left[\frac{1}{\delta_1}\right])\int_0^t \chi\left(W_n(s)\right)ds \qquad (9)$$

for all $t \in [0, T]$. Because W_n is decreasing for n, we have

$$\sigma(t) = \lim_{n\to\infty} \chi(W_n(t))$$

for all $t \in [0, T]$. From (9), we have

$$\sigma(t) \leq 2M_0K_0\left[\frac{1}{\delta_1}\right]\int_0^t \left(\sigma(s)\right)ds$$

fort $\in [0, T]$, which implies that $\sigma(t) = 0$ for all $t \in [0, T]$. By Lemma 2.3, we know that $\lim_{n\to\infty} \chi(W_n(t)) = 0$. Using Lemma 2.1 we know that $W = \cap_{n=1}^{\infty} W_n$ is convex compact and nonempty in $C([0, T]; X)$and $F(W) \subset W$. By the famous Schauder's fixed point theorem, there exist at least one mild solution u of the initial value problem $(1) - (2)$, where $u \in W$ is a fixed point of the continuous map F.

Remark 3.2. If the functions f and g are compact (see e.g. [4, 5, 14]), then $(H2)$ is automatically satisfied. In some of the early related results in references and above results, it is supposed that the map h is uniformly bounded. We indicate here that this condition can be released. In fact, if h is compact, then it must be bounded on bounded set. Here we give an existence result under growth condition of f, when g is not uniformly bounded. Precisely, we replace the assumptions$(H2)$ by

$(H5)$ There exist a functions $p \in L(0,T;R^+)$ andq $\in L(0,T;R^+)$. The Constants $b_1, b_2 > 0$ such that

$$||f(t,u)|| \leq p(t)b_1||u||$$

for a.e $t \in [0,T]$ and all $u \in C([0,T];X)$.

Theorem: 3.2. Suppose that the assumptions $(H1)-(H5)$ are satisfied, then the equation $(1) - (2)$ has at least one mild solution if

$$\lim_{r\to\infty} \sup \frac{M_0}{r}\left(\varphi(r) + rT\left[\frac{pb_1}{\delta_1}\right]\right) < 1 . \qquad (10)$$

Where $\varphi(r) = \sup\{||g(u)||, ||u|| \leq r\}$.

Proof.The inequality (10) implies that there exist a constant $r > 0$ such that
$M_0\left(\varphi(r) + rT\left[\frac{pb_1}{\delta_1}\right]\right) < r.$
Just as in the proof of Theorem 3.1, let $W_0 = \{u \in C([0,T];X), \left||u(t)|\right| \leq r\}$
and $W_1 = \overline{conv}FW_0$. Then for any $u \in W_1$, we have

$$\begin{aligned} \left||u(t)|\right| &\leq ||U_u(t,0)g(u)|| + \int_0^t U_u(t,s)\,[f\left(s,u(\alpha(s))\right)]ds \\ &\leq M_0\varphi(r) + M_0\left[\int_0^t \mathrm{p}(s)b_1||u(\alpha(s)||ds\right] \\ &\leq M_0\varphi(r) + M_0\left(\left[\frac{p(t)b_1rT}{\delta_1}\right]\right) \end{aligned}$$

$$||u(t)|| \leq M_0\left(\varphi(r) + rT\left[\frac{pb_1}{\delta_1}\right]\right) < r$$

for$t \in [0,T]$. It means that $W1 \subset W0$. So can completethe proof similarly to Theorem 3.1.

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