The Generalization of Graph Operations and Their Local Metric Dimension

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Abstract
Let G be a connected graph with vertex set V(G) and $W = \{w_1, w_2, \ldots, w_m\} \subset V(G)$. The representation of a vertex $v \in V(G)$ with respect to W is the ordered m-tuple $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_m))$ where $d(v, w)$ represents the distance between vertices v and w. The set W is called a resolving set for G if every vertex of G has a distinct representation with respect to W. A resolving set containing a minimum number of vertices is called basis for G. The metric dimension of G, denoted by$dim(G)$, is the number of vertices in a basis of G.The set W is called a local resolving set for G if every twoadjacent vertices of G have a distinct representationanda minimum localresolving set iscalled alocal basisofG. The cardinalityofa local basis ofGiscalled thelocalmetricdimensionofG,denotedbydim$_l$(G). The comb product and the corona product are non-commutative operations in graph, but these operations can be commutative with respect to the local metric dimension for some graphs with certain conditions. In thispaper, we determine the local metric dimension of the most generalized comb and corona products of graphs. Futhermore, we determine the commutative characterization of comb and corona products with respect to the local metric dimension.

Keywords - *resolving set, basis, local basis, local metric dimension,the most generalizedcomb and corona products of graph,commutative characterization.*

I. INTRODUCTION

Let *G*be a finite, simple, andconnected graph. The vertex and edge sets of graph *G* are denoted by *V*(*G*)and *E*(*G*), respectively. The distancebetween vertices *v* and *w* in *G*, denoted by $d(v, w)$, is the length of the shortest path between them.For the ordered set $W = \{w_1, w_2, \ldots, w_m\} \subseteq V(G)$, and a vertex$v \in V(G)$,the representation of v with respect to W is the*m*-tuple,$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_m))$.The set W is called a *resolving set*of$G$ if every vertex of G has a distinct representation with respect to *W*. A minimum resolving set$W$of graph G is called a *basis*of$G$. Thecardinalityofa basis iscalled*metric dimension* of G, denoted by dim(*G*) [1].The set W is called a*local resolving set*of$G$ if everytwoadjacent vertices of G have a distinct representationwithrespecttоW, that isif$u, v \in V(G)$ such that$uv \in E(G)$) then$r(u|W) \neq r(v|W)$.Alocal resolving set of G with minimum cardinality is called a *local basis* of *G*, and the cardinality of a local basis of *G*is called the*local metric dimension* of *G*, denoted by dim$_l$(*G*).
Godsil and McKay [3]defined the rooted product graph as follows. Let *G* be a graph on *n* vertices and $\mathcal{H}$be a sequence of *n*rooted graphs H_1, H_2, H_3, ..., H_n. The rooted product graph of *G* by $\mathcal{H}$ denoted by $Go\mathcal{H}$is a graph obtained by identifying the root of H_i with the *i*-th vertex of *G*.Frucht and Harary [2] defined the corona product graph. The corona graph,$G \odot H$, of two graphs *G* and *H* is obtained by taking one copy of *G* and $|V(G)|$ copies of *H* and then joining by edge the *i*-th vertex of *G* to every vertex in the*i*-th copy of *H*. In [6], Rodriguez et al. generalized the corona product $G \odot \mathcal{H}$, where$\mathcal{H}$is a sequence of n graphsH_1, H_2, H_3, ..., H_n, H_i and H_j may not be isomorphic.Saputro et al. [7] studied the metric dimension of the comb productgraph GoH, which is a special case of a rooted product graph.Susilowati et al. [10] have left the open problem on the metric dimension of $(k_1, k_2, k_3, \ldots, k_n)$-comb and $(k_1, k_2, k_3, \ldots, k_n)$-corona of graph *G* of order *n* and *n* sequence of graphs $\mathcal{H}$. In this paper, we determine the local metric dimension of $(k_1, k_2, k_3, \ldots, k_n)$-comb and $(k_1, k_2, k_3, \ldots, k_n)$-corona of graph *G* of order *n* and *n* sequence of graphs $\mathcal{H}$ and the commutative characterization of comb and corona product with respect to local metric dimension.

Rodriguez et al. [6] observed the local metric dimension of rooted product graph as follow:

Theorem 1.1. [6] *Let G be a connected labelled graph of order $n \geq 2$ and let $\mathcal{H}$ be a sequence of n connected bipartite graphs H_1, H_2, H_3, ..., H_n. Then, for any rooted product graph $Go\mathcal{H}$*,dim$_l$($Go\mathcal{H}$)=dim$_l$(*G*).

Theorem 1.2. [6] *Let G be a connected labelled graph of order $n \geq 2$ and let $\mathcal{H}$ be a sequence of n connected non-bipartite graphs H_1, H_2, H_3, ..., H_n. Then, for any rooted product graph $Go\mathcal{H}$*,dim$_l$($Go\mathcal{H}$)= $\sum_{j=1}^{n}(\dim_l(H_j) - \alpha_j)$, *where* $\alpha_j = 1$ *if the root of* H_j *belongs to a local basis of* H_j *and* $\alpha_j = 0$ *otherwise.*

Susilowati et al [10] defined the generalized comb product of graphs $G o_k \mathcal{H}$ and the generalized corona product of graphs $G \odot_k \mathcal{H}$. Futhermore, Susilowati et al [10] determined the metric dimension of $G\ o_k \mathcal{H}$ and $G \odot_k \mathcal{H}$. Rodriguez et al. [5] observed the local metric dimension of corona product graphs, as bellow.

Theorem 1.3. [5] *Let H be a non empty graph. The following statements hold.*
(i). *If the vertex of* K_1 *does not belong to any local basis for* $K_1 + H$, *then for any connected graph G of order n,*

$$\dim_l(G \odot H) = n \dim_l(K_1 + H).$$

(ii). *If the vertex of* K_1 *belongs to a local basis for* $K_1 + H$, *then for any connected graph G of order n* ≥ 2,

$$\dim_l(G \odot H) = n(\dim_l(K_1 + H) - 1).$$

Theorem 1.4. [6] *Let G be a connected labeled graph of order n* ≥ 2 *and let* $\mathcal{H}$ *be a sequence of n non emptygraphs* $H_1, H_2, H_3, \ldots, H_n$. *Then, for any corona product graph* $G \odot \mathcal{H}$,

$$\dim_l(G \odot \mathcal{H}) = \sum_{j=1}^{n}(\dim_l(K_1 + H_j) - \alpha_j),$$

where $\alpha_j = 1$ *if the vertex of* K_1 *belongs to a local basis of* $K_1 + H_j$ *and* $\alpha_j = 0$ *otherwise.*

Okamoto et al. [4] discovered the characterization of local metric dimension for some graphs. Meanwhile, Rodriguez et al. [6] and Susilowati et al. [8] observed the local metric dimension of rooted product graph.Susilowati et al [9] also determined the local metric dimension of $G\ o_k \mathcal{H}$ and $G \odot_k \mathcal{H}$.

In this paper, we define the most generalized of comb product of graphs ($G\ o_{k_1,k_2,k_3,\ldots k_n} H$), rooted product of graphs ($G\ o_{k_1,k_2,k_3,\ldots k_n} \mathcal{H}$)and the most generalized corona product of graphs ($G \odot_{k_1,k_2,k_3,\ldots k_n} H$ and $G \odot_{k_1,k_2,k_3,\ldots k_n} \mathcal{H}$), where$n = |V(G)|$. Futhermore, weanalyse the metric dimension and local metric dimension of$G\ o_{k_1,k_2,k_3,\ldots k_n} H, G\ o_{k_1,k_2,k_3,\ldots k_n} \mathcal{H}, G \odot_{k_1,k_2,k_3,\ldots k_n} H$ and $G \odot_{k_1,k_2,k_3,\ldots k_n} \mathcal{H}$. We also formulate the necessary and sufficient conditions such that the local metric dimension of comb product graphs has the equal value, even though the position of graph that operated is exchanged. Likewise for the corona product graphs.

II.THE LOCAL METRIC DIMENSION OF THE MOST GENERALIZED OF COMB PRODUCT GRAPHS

Let G be a labelled graph on n vertices and $\mathcal{H}$be a sequence of n rooted graphs $H_1, H_2, H_3, \ldots, H_n$. The $\boldsymbol{k_1, k_2, k_3, \ldots k_n}$**-rooted**product graph of G by $\mathcal{H}$ denoted by $\boldsymbol{G\ o_{k_1,k_2,k_3,\ldots k_n} \mathcal{H}}$ is obtained by taking one copy of G andk_i copies of H_i for every i= 1, 2, ...,n, that are$H_{11}, H_{12}, H_{13}, \ldots, H_{1k_1}, H_{21}, H_{22}, H_{23}, \ldots, H_{2k_2}, H_{31}, H_{32}, H_{33}, \ldots, H_{3k_3}, \ldots, H_{n1}, H_{n2}, H_{n3}, \ldots, H_{nk_n}$ and grafting the rootofH_{ij}, $j = 1, 2, 3, \ldots, k_j$with the i-th vertex of G. If o_{js} is the root of H_{js}, for $s = 1, 2, \ldots, k$, then$o_{js} = o_j$ in the graph $\boldsymbol{G\ o_{k_1,k_2,k_3,\ldots k_n} \mathcal{H}}$, for $s = 1, 2, \ldots, k_j$. If $H_i \cong H$ for every $i = 1, 2, 3, \ldots, n$, then we get $G\ o_{k_1,k_2,k_3,\ldots k_n} \mathcal{H} \cong G\ o_{k_1,k_2,k_3,\ldots k_n} H$. In other words, $G\ o_{k_1,k_2,k_3,\ldots k_n} H$ is the special case of $G\ o_{k_1,k_2,k_3,\ldots k_n} \mathcal{H}$.

Susilowati et al. [8] described the properties of rooted product graphs as the following lemma and observation.

Observation 2.1. [8] Let *G be a labelled graph of order n* ≥ 2 *and*$\mathcal{H}$ *be a sequence of n connected graphs* H_j, *j = 1, 2, 3, ...,n. In the rooted product graph*$G o \mathcal{H}$, *if every* H_j *is connected bipartite graph then every two adjacent vertices in* H_j *have distinct distance to the root of* H_j *and to all vertices in* $G o \mathcal{H}$.

Lemma 2.2. [8]Let *G be a labelled graph of order n* ≥ 2 *and*$\mathcal{H}$ *be a sequence of n connected graphs* H_j, *j = 1, 2, ..., n. In the rooted product graph*$G\ o\ \mathcal{H}$,*if* o_j *is the root of*H_j, and U_j*is local basis of* H_j, *then the following statements hold.*

(i). *If* $o_j \in U_j$ *then there are two adjacent vertices x,y in* H_j *such that*$r(x|S) = r(y|S)$*for every*$S \subset V(H_j)$, $|S| \leq |U_j| - 2$.

(ii). *If* $o_j \notin U_j$*then there are two adjacent vertices x,y in* H_j *such that*$r(x|S) = r(y|S)$*for every*$S \subset V(H_j)$, $|S| \leq |U_j| - 1$.

Using Theorems 1.1. and 1.2 respectively, we get the corollaries 2.3 and 2.4 respectively, as below.

Corollary 2.3. *Let G be a labelled connected graph of order n* ≥ 2. *Let H be a connected graph and* $\mathcal{H}$ *be a sequence of n connected bipartite rooted graphs* $H_1, H_2, H_3, \ldots, H_n$. *Then*

$$\dim_l(G\ o_{k_1,k_2,k_3,\ldots k_n} \mathcal{H}) = \dim_l(G\ o_{k_1,k_2,k_3,\ldots k_n} H) = \dim_l(G).$$

Proof: Let Gbe a labelled connected graph of order $n \geq 2$ and let$\mathcal{H}$be a sequence of $n$connected bipartite rooted graphs $H_1, H_2, H_3, \dots, H_n$.Let $H_{11}, H_{12}, H_{13}, \dots, H_{1k_1}, H_{21}, H_{22}, H_{23}, \dots, H_{2k_2}, H_{31}, H_{32}, H_{33}, \dots, H_{3k_3}, \dots, H_{n1}, H_{n2}, H_{n3}, \dots, H_{nk_n}$ are the k_i copies of H_i for every $i = 1, 2, 3, \dots, n$.Let o_{js} is the root of H_{js}, for s = 1, 2, ..., k_j, choose$W$ = local basis of G.

Take any two adjacent vertices x, y in H_{js}, j = 1, 2, ..., n ; s = 1, 2, ..., k_j.BecauseH_{js}connected bipartite, by Observation 2.1, we get $d(x|z) \neq d(y|z)$ for every$z \in V(G\ \mathrm{o}_{k_1,k_2,k_3,\dots k_n}\mathcal{H})$.Therefore $r(x|W) = r(y|W)$.Take any two adjacent roots o_{is}, o_{js} in $G\ \mathrm{o}_{k_1,k_2,k_3,\dots k_n}\mathcal{H}$. Because W = local basis of G then $r(o_{is}|W) \neq) r(o_{js}|W)$ and W is a local basis of $G\ \mathrm{o}_{k_1,k_2,k_3,\dots k_n}\mathcal{H}$. So $\dim_l(G\ \mathrm{o}_{k_1,k_2,k_3,\dots k_n}\mathcal{H}) = \dim_l(G)$.The same reason for $\dim_l(G\ \mathrm{o}_{k_1,k_2,k_3,\dots k_n}H) = \dim_l(G)$.■

Corollary 2.4. *Let G be a connected labelled graph of order n and let $\mathcal{H}$ be a sequence of n connected non-bipartite rooted graphs of order at least two*H_1, H_2, H_3, ..., H_n.*If o_j is the root of H_j for every j=1, 2,...,n, then*

$\dim_l(G\ \mathrm{o}_{k_1,k_2,k_3,\dots k_n}\mathcal{H}) = \sum_{j=1}^{n} k_j\,(\dim_l(H_j) - \alpha_j)$

where $\alpha_j = 1$ if o_j belongs to a basis of H_j and $\alpha_j = 0$ otherwise.

Proof: Let G be a connected *labelled*graph of order $n \geq 2$ and let$\mathcal{H}$be a sequence of the connected non bipartite rooted graphs H_1, H_2, H_3, .. ., H_n.Let $H_{11}, H_{12}, H_{13}, \dots, H_{1k_1}, H_{21}, H_{22}, H_{23}, \dots, H_{2k_2}, H_{31}, H_{32}, H_{33}, \dots, H_{3k_3}, \dots, H_{n1}, H_{n2}, H_{n3}, \dots, H_{nk_n}$ are the k_i copies of H_i for every $i = 1, 2, 3, \dots, n$.Let o_{js} is the root of H_{js} and W_{js} is a local basis of $H_{js}$for $i = 1, 2, 3, \dots, n$.;s = 1, 2, ..., k_j.Choose $W = \bigcup_{j=1}^{n}(\bigcup_{s=1}^{k}(W_{js} - \{o_{js}\}))$.

Take any two adjacent vertices x, y in $G\ \mathrm{o}_{k_1,k_2,k_3,\dots k_n}\mathcal{H}$.By Lemma 2.2,we can proof that$W$ is a minimum local resolving set of $G\ \mathrm{o}_{k_1,k_2,k_3,\dots k_n}\mathcal{H}$ and $|W| = \sum_{j=1}^{n} k_j\,(\dim_l(H_j) - \alpha_j)$, where $\alpha_j = 1$ if o_j belongs to a basis of H_j and $\alpha_j = 0$ otherwise.■

By Corollaries 2.3 and 2.4 we get Corollary 2.5 and Corollary 2.6 respectively, as below.

Corollary 2.5. *Let G and H be connected graphs. If H be a bipartite graph, then*

$$\dim_l(G\mathrm{o}_{k_1,k_2,k_3,\dots k_n}H) = \dim_l(G).$$

Corollary 2.6.*Let G be a connected graph of order n, H be a connected non bipartite graph of order at least 2, and o is a grafting vertex. Then*

$$\dim_l(G\mathrm{o}_{k_1,k_2,k_3,\dots k_n}H) = \begin{cases} \sum_{j=1}^{n} k_j\,(\dim_l(H) - 1), & \text{if } o \text{ belongs to local basis of } H \\ \sum_{j=1}^{n} k_j\,(\dim_l(H)), & \text{otherwise} \end{cases}$$

Theorem 2.7. *Let G be a connected labelled graph of order $n \geq 2$, and let $\mathcal{H}$ be a sequence of the combined of n connected non-bipartite H_1, H_2, ..., H_s and bipartite graphs H_{s+1}, H_{s+2}, ..., H_n, and o_j is the root ofH_j. Then*

$$\dim_l(G\mathrm{o}_{k_1,k_2,k_3,\dots k_n}\mathcal{H}) \begin{cases} = \sum_{j=1}^{s} k_j\,(\dim_l(H_j) - \alpha_j), & \text{for } G = C_n, n \text{ odd}, s > 1 \\ & \text{or } G \text{ bipartite or } G = K_n, s = n-1 \\ = \sum_{j=1}^{s} k_j\,(\dim_l(H_j) - \alpha_j) + 1, & \text{for } G = C_n, n \text{ odd}, s = 1 \\ = \sum_{j=1}^{s} k_j\,(\dim_l(H_j) - \alpha_j) + \dim_l(G) - s, & \text{for } G = K_n, s < n-1 \\ < \sum_{j=1}^{s} k_j\,(\dim_l(H_j) - \alpha_j) + n - s - 1, & \text{otherwise} \end{cases}$$

where$\alpha_j = 1$ if the root of H_j belongs to a local basis of H_j and $\alpha_j = 0$ otherwise.

Proof: Case 1:for $G = C_n, n$ odd, $s > 1$ or G bipartite or $G = K_n, s = n - 1$ .Choose $W = \bigcup_{j=1}^{s}(\bigcup_{l=1}^{k_j}(U_{jl} - \{o_{jl}\}))$, so $|W| = \sum_{j=1}^{s} k_j\,(\dim_l(H_j) - \alpha_j)$.

By Lemma 2.2. we can see that $W = \bigcup_{j=1}^{s}(\bigcup_{l=1}^{k_j}(U_{jl} - \{o_{jl}\}))$ is a minimum local resolving set of $G\mathrm{o}_{k_1,k_2,k_3,\dots k_n}\mathcal{H}$and $\dim_l(G\mathrm{o}_{k_1,k_2,k_3,\dots k_n}\mathcal{H}) = \sum_{j=1}^{s} k_j\,(\dim_l(H_j) - \alpha_j)$.

Case 2: for $G = C_n, n$ odd, $s = 1$.Choose $= \bigcup_{j=1}^{s}(\bigcup_{l=1}^{k_j}(U_{jl} - \{o_{jl}\})) \cup \{z\} = \bigcup_{l=1}^{k_j}(U_{1l} - \{o_{1l}\}) \cup \{z\}$, $z \in H_{jl}$ for any$j = s + 1$, s +2, ..., n; l = 1, 2, ..., k_j and $z \neq o_{1l}$. We can proof that W is a minimum local resolving set of $G\mathrm{o}_{k_1,k_2,k_3,\dots k_n}\mathcal{H}$and $\dim_l(G\mathrm{o}_{k_1,k_2,k_3,\dots k_n}\mathcal{H}) = \sum_{j=1}^{s} k_j\,(\dim_l(H_j) - \alpha_j) + 1$.

Case 3: for $G = K_n, s < n - 1$. Choose

$W = \bigcup_{j=1}^{s}(\bigcup_{l=1}^{k_j}(U_{jl} - \{o_{jl}\})) \cup \{u_{jl} | u_{jl} \neq o_{jl}, j = s+1, s+2, \dots, n-1, l = 1,2,3, \dots k_j\}$,Without lossofgenerality, let$s = n - 2$, it meansthatH_{jl}, j = 1, 2, ..., n – 2 are non bipartite graphs and $H_{(n-1)l}$, H_{nl} are bipartite graphs and

$$W = \cup_{j=1}^{n-2}(\cup_{l=1}^{k_j}(U_{jl} - \{o_{jl}\})) \cup \{u_{(n-1)l} | u_{(n-1)l} \neq o_{(n-1)l}, l = 1,2,3, \dots k_j\}.$$

We can proof that W is a minimum local resolving set of $Go_{k_1,k_2,k_3,\dots k_n}\mathcal{H}$and

$\dim_l(Go_{k_1,k_2,k_3,\dots k_n}\mathcal{H}) = \sum_{j=1}^{s} k_j(\dim_l(H_j) - \alpha_j) + \dim_l(G) - s.$

Case 4: For G is otherwise, $\dim_l(Go_{k_1,k_2,k_3,\dots k_n}\mathcal{H}) < \sum_{j=1}^{s} k_j(\dim_l(H_j) - \alpha_j) + n - s - 1$, it is obvious because$K_n$ is the graph with the biggest local metric dimension respect to the its order. ■

III.THE LOCAL METRIC DIMENSION OF THE MOST GENERALIZED OF CORONA PRODUCT GRAPHS

Let G be a labelled graph on n vertices and $\mathcal{H}$be a sequence of n rooted graphs H_1, H_2, H_3, ..., H_n. The $\boldsymbol{k_1, k_2, k_3, \dots k_n}$**-corona**product graph of G by $\mathcal{H}$ denoted by$G\odot_{k_1,k_2,k_3,\dots k_n}\mathcal{H}$is obtained by taking one copy of Gandk_i copies of H_i for every i= 1, 2, ...,n, that are $H_{11}, H_{12}, H_{13}, \dots, H_{1k_1}, H_{21}, H_{22}, H_{23}, \dots, H_{2k_2}, H_{31}, H_{32}, H_{33}, \dots, H_{3k_3}, \dots, H_{n1}, H_{n2}, H_{n3}, \dots, H_{nk_n}$ andthenjoiningb yedgethe$i$-thvertexof$G$ to every vertex in j-th copy of H_i, j = 1, 2, 3, ..., k_j. If $H_i \cong H$ for every $i = 1, 2, 3, \dots, n$, then we get $G \odot_{k_1,k_2,k_3,\dots k_n}\mathcal{H} \cong G \odot_{k_1,k_2,k_3,\dots k_n} H$. In other words, $G \odot_{k_1,k_2,k_3,\dots k_n} H$ is the special case of $G \odot_{k_1,k_2,k_3,\dots k_n}\mathcal{H}$.

Lemma 3.1. *Let G be a connected nontrivial labelledgraph. IfH be an empty graphand $\mathcal{H}$ be a sequence of n empty graphs H_1, H_2, H_3, ..., H_n,then*

$$\dim_l(G\odot_{k_1,k_2,k_3,\dots k_n} H) = \dim_l(G\odot_{k_1,k_2,k_3,\dots k_n}\mathcal{H}) = \dim_l(G).$$

Proof: Let$H$ be an empty graph and $\mathcal{H}$be a sequence of n empty graphs H_1, H_2, H_3, ..., H_n. Thenthere are no edge in H and $\mathcal{H}$.In graph $G\odot_{k_1,k_2,k_3,\dots k_n} H$ and $G\odot_{k_1,k_2,k_3,\dots k_n}\mathcal{H}$ respectively, every vertex in H and $\mathcal{H}$adjacents to one vertex only in G. Therefore, the local metric dimension of $G\odot_{k_1,k_2,k_3,\dots k_n} H$ and $G\odot_{k_1,k_2,k_3,\dots k_n}\mathcal{H}$depend on local metric dimension of$G$only. ■

Using Theorem 1.3. and 1.4 respectively, we get the Corollaries 3.2 and 3.3 respectively, as bellow.

Corollary 3.2. *Let H be a non empty graph. The following statements hold.*
(i).If the vertex of K_1does not belong to any local basis for $K_1 + H$, then for any connected graph G of order n,

$$\dim_l(G\odot_{k_1,k_2,k_3,\dots k_n} H) = \sum_{i=1}^{n} k_i(\dim_l(K_1 + H)).$$

(ii).If the vertex of K_1 belongs to a local basis for $K_1 + H$, then for any connected graph G of order n ≥ 2,

$$\dim_l(G\odot_{k_1,k_2,k_3,\dots k_n} H) = \sum_{i=1}^{n} k_i(\dim_l(K_1 + H) - 1).$$

Corollary 3.3. *Let G be a connected labelled graph of order n ≥ 2, and $\mathcal{H}$ be a sequence of n non emptygraphsH_1, H_2, H_3, ..., H_n. Then*

$$\dim_l(G\odot_{k_1,k_2,k_3,\dots k_n}\mathcal{H}) = \sum_{i=1}^{n} k_i(\dim_l(K_1 + H_i) - \alpha_i)$$

where $\alpha_i = 1$ if the vertex of K_1 belongs to a local basis of $K_1 + H_j$ and $\alpha_j = 0$ otherwise.

Proof: Let$V(G) = \{v_1, v_2, v_3, \dots, v_n,\}$,$B_i$isa local basis of H_i and B_{ij} is a basis of $< v_i > + H_{ij}$, i = 1, 2, ..., n;j = 1, 2, ..., k_i. So $B_{ij} = B_i$ for j = 1, 2, ..., k_i. Choose $W = \cup_{i=1}^{n}(\cup_{j=1}^{k_i}(B_{ij} - \{v_i\}))$.Because$< v_i > + H_{ij} \approx K_1 + H_iso|W| = \sum_{i=1}^{n} k_i(\dim_l(K_1 + H_i) - 1)$if $v_i$is an element of a local basis of$K_1 + H_i$ and $|W| = \sum_{i=1}^{n} k_i(\dim_l(K_1 + H_i))$if v_i is not an element of a local basis of $K_1 + H_i$.Futhermore, we can prove that W is a local basis of $G\odot_{k_1,k_2,k_3,\dots k_n}\mathcal{H}$. ■

IV. COMMUTATIVE CHARACTERIZATION OF COMB AND CORONA PRODUCTS GRAPHS WITHRESPECT TO THELOCAL METRIC DIMENSION

An operation * defined on two graphs is said commutative if A*B≅B*$A$for every graph $A$and B.An operation * defined on two graphs G and H is said commutative with respect to local metric dimension if $\dim_l(G*H) = \dim_l(G*H)$, denoted by $(G*H) \cong_{\dim l}(H*G)$ [9].
In this section, we present commutative characterization of comb and corona products graphs with respect to the local metric dimension.

Theorem 4.1.*Let G and$H$be connected bipartite graphs of order at least two.Then*

$$\dim_l(G) = \dim_l(H) \textit{ if and only if } (G\mathbf{o}H) \cong_{\dim l} (H\mathbf{o}G)$$

Proof: Let $G$and$H$be connected bipartite graphs. Let$(G\mathbf{o}H) \cong_{\dim l} (H\mathbf{o}G)$.It means that $\dim_l(G\mathbf{o}H) = \dim_l(H\mathbf{o}G)$.By Theorem 1.1,we get$\dim_l(G) = \dim_l(H)$.

Conversely, let $\dim_l(G) = \dim_l(H)$. By Theorem 1.1, we get $\dim_l(G) = \dim_l(G\mathbf{o}H)$ and $\dim_l(H) = \dim_l(H\mathbf{o}G)$. Therefore $\dim_l(G\mathbf{o}H) = \dim_l(H\mathbf{o}G)$. So $(G\mathbf{o}H) \cong_{\dim l} (H\mathbf{o}G)$.■

For the case of non bipartite graphs, the formula of commutative characterization of generalized comb product with respect to the local metric dimension is presented base on existence of grafting vertex, whether or not element of a local basis of graph operated.

Theorem 4.2. *Let G and H be connected graphs of order at least three. Let G and H be non bipartite graphs. If the grafting vertex of $G\mathbf{o}H$ belongs to a local basis of H and the grafting vertex of $H\mathbf{o}G$ belongs to a local basis of G, then*

$$|V(G)|\dim_l(H-1) = |V(H)|\dim_l(G-1)\textit{\textbf{ if and only if }}(G\mathbf{o}H) \cong_{\dim l} (H\mathbf{o}G)$$

Proof: Let G and H be connected graphs of order at least three. Let G and H be non bipartite graphs. Let the grafting vertex of $G\mathbf{o}H$ belongs to a local basis of H and the grafting vertex of $H\mathbf{o}G$ belongs to a local basis of G. Let $(G\mathbf{o}H) \cong_{\dim l} (H\mathbf{o}G)$. Theorem 1.2, we get $\dim_l$ $(G\mathbf{o}H)$ = $|V(G)|(\dim_l(H)-1)$ and $\dim_l(H\mathbf{o}G) = |V(H)|(\dim_l(G)-1)$.
Therefore $|V(G)|(\dim_l(H)-1)) = |V(H)|(\dim_l(G)-1)$.
So $|V(G)|(\dim_l(H-1) = |V(H)|(\dim_l(G)-1)$.
Conversely, let $|V(G)|(\dim_l(H)-1) = |V(H)|(\dim_l(G)-1)$. Then

$$|V(G)|(\dim_l(H)-1) = |V(H)|(\dim_l(G)-1).$$

Therefore $\dim_l(G\mathbf{o}H)$ = $\dim_l(H\mathbf{o}G)$. ■

For the case grafting vertex does not belong to a local basis of graph operated, given below.

Theorem 4.3. *Let G and H be connected graphs of order at least three. Let G and H be non bipartite graphs. If the grafting vertex of $G\mathbf{o}H$ does not belong to a local basis of H and the grafting vertex of $H\mathbf{o}G$ does not belong to a local basis of G, then*

$$|V(G)|\dim_l(H) = |V(H)|\dim_l(G)\textit{\textbf{ if and only if }}(G\mathbf{o}H) \cong_{\dim l} (H\mathbf{o}G).$$

Proof: Let G and H be connected graphs of order at least three. Let G and H be non bipartite graphs. Let the grafting vertex of $G\mathbf{o}H$ does not belong to a local basis of H and the grafting vertex of $H\mathbf{o}G$ does not belong to a local basis of G. Let $(G\mathbf{o}H) \cong_{\dim l} (H\mathbf{o}G)$. By Theorem 1.2, we get $\dim_l(G\mathbf{o}H) = |V(G)|(\dim_l(H))$ and $\dim_l(H\mathbf{o}G) = |V(H)|(\dim_l(G))$.
Therefore $|V(G)|(\dim_l(H)) = |V(H)|(\dim_l(G))$. So $|V(G)|\dim_l(H) = |V(H)|\dim_l(G)$.
Conversely, let $|V(G)|(\dim_l(H) = |V(H)|(\dim_l(G))$. Then $\dim_l(G\mathbf{o}H)$ = $\dim_l(H\mathbf{o}G)$. In other words $(G\mathbf{o}H) \cong_{\dim l} (H\mathbf{o}G)$■

In the next theorems, we present the commutative characterization of generalized corona product with respect to local metric dimension.

Theorem 4.4. *Let G and H be non empty connected graphs. If the vertex of K_1 does not belong to a local basis of $K_1 + H$ and $K_1 + G$, then*

$$|V(G)|(\dim_l(K_1+H)) = |V(H)|\dim_l(K_1+G))\textit{\textbf{ if and only if }}(H \odot G) \cong_{\dim l} (G \odot H)$$

Proof: Let G and H be non empty connected graphs. Let the vertex of K_1 does not belong to a local basis of $K_1 + H$ and $K_1 + G$. Let $(G \odot H) \cong_{\dim l} (H \odot G)$. Based on Theorem 1.3.(i), we get
$\dim_l(G \odot H) = |V(G)|\ (\dim_l(K_1+H))$ and $\dim_l(G\odot_k H) = |V(G)|\ (\dim_l(K_1+H))$.
Therefore $|V(G)|\ (\dim_l(K_1+H)) = |V(H)|\ (\dim_l(K_1+G))$.
So $|V(G)|\dim_l(K_1+H) = |V(H)|\dim_l(K_1+H)$.
Conversely, let $|V(G)|\dim_l(K_1+H) = |V(H)|\dim_l(K_1+H)$. Then
Based on Theorem 1.3.(i), we get $\dim_l(G \odot H) = \dim_l(H \odot G)$.
In other words $(G \odot H) \cong_{\dim l} (H \odot G)$. ■

Theorem 4.5. *Let G and H be non empty connected graphs of order at least two. If the vertex of K_1 belongs to a local basis of $K_1 + H$ and $K_1 + G$. Then*

$$|V(G)|(\dim_l(K_1+H)-1) = |V(H)|\dim_l((K_1+G)-1)$$
$$\textit{\textbf{if and only if }}(H \odot G) \cong_{\dim l} (G \odot H).$$

Proof: Let G and H be non empty connected graphs of order at least two. Let the vertex of K_1 belongs to a local basis of $K_1 + H$ and $K_1 + G$. Let $(G \odot H) \cong_{\dim l} (H \odot G)$. By Theorem 1.3.(ii), we get,

$\dim_l(G \odot H) = |V(G)|\ (\dim_l(K_1+H)-1)$ and $\dim_l(H \odot G) = |V(H)|\ (\dim_l(K_1+G)-1)$.

Therefore $|V(G)|\ (\dim_l(K_1+H)-1) = |V(H)|\ (\dim_l(K_1+G)-1)$.

Conversely, let$|V(G)|$ $(\dim_l(K_1+H)-1) = |V(H)|$ $(\dim_l(K_1+G)-1)$.Then, byTheorem 1.3.(ii), we get $\dim_l(G\odot H)=\dim_l(H\odot G)$. In other words$(G\odot H)\cong_{\dim l}(H\odot G)$. ■

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