Reduction of Quasi-Lattices to Lattices

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Abstract

Quasi-lattices are introduced in terms of 'join' and 'meet' operations. It is observed that quasi-lattices become lattices when these operations are associative and when these operations satisfy 'modularity' conditions. A fundamental theorem of homomorphism proved in this article states that a quasi-lattice can be mapped onto a lattice when some conditions are satisfied.

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1 Introduction

The concept of a minimal upper bound is not widely known. A lattice is a partially ordered set (poset) in which any two elements have a least upper bound and a greatest lower bound. A quasi-lattice is a poset in which any two elements have a minimal upper bound and a maximal lower bound. Every quasi-lattice is a lattice. This article tries to establish fundamental facts about quasi-lattices. But, it finds that associativity of 'meet' and

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'join' operations of quasi-lattices is a unique property of lattices. Similarly it is established that 'modularity' is also a unique property of lattices. A fundamental theorem of homomorphism found in this article also reduces quasi-lattices into lattices. The books [3] and [2] are referred to fundamental definitions and properties for posets and lattices. Although there are many recent articles (see, for example [4, 5, 6]) the results of these articles will not be extended to quasi-lattices, because quasi-lattices reduce to lattices when some fundamental properties are assumed.

A partial order $\leq$ on a non empty set P is a relation that is reflexive, anti-symmetric and transitive. A poset $(P, \leq)$ is a non empty set P with a partial order $\leq$. An element a in a partially ordered set $(P, \leq)$ is a maximal lower bound of a non empty subset A of P if $a \leq x$, $\forall x \in A$, and if there is no element d in P such that $a < d \leq x$, $\forall x \in A$.Dually a minimal upper bound is defined. A partially ordered set $(P, \leq)$ is called quasi-lattice, if any two elements of P have a minimal upper bound and a maximal lower bound. However, two elements in a quasi-lattice may have more than one maximal lower bound and may have more than one minimal upper bound. Let us use the notations $x \wedge y$ and $x \vee y$ to denote some (particular) maximal lower bound and some minimal upper bound of x and y, respectively, in a partially ordered set.

Example 1.1 *The Hasse diagram given in the Figure 1 represents a quasi-lattice. In this diagram the point $x \vee (y \vee z)$ represents another minimal upper bound of $\{x,y\}$ in addition to $x \vee y$. So, it is not a lattice. In this quasi-lattice, $(x \vee y) \vee z$ can never take " the form" $x \vee (y \vee z)$. So, associativity fails to be true.*

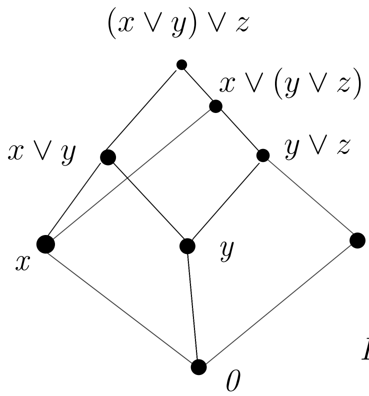

Figure 1

2 Associative quasi-lattices

It would be difficult to derive many results for quasi-lattices, when associativity is not assumed.

Definition 2.1 *A quasi-lattice $(P, \leq)$ is called an associative lattice, if*

(i) $a \vee (b \vee c) = (a \vee b) \vee c$, *and*

(ii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ *hold for every* $a, b, c \in P$.

Here (i) means that if a_1 is a minimal upper bound of $\{b, c\}$ and if a_2 is a minimal upper bound of $\{a, a_1\}$, then there is a minimal upper bound a_3 of $\{a, b\}$ such that a_2 is a minimal upper bound of$\{a_3, c\}$ and similarly; if b_1 is a minimal upper bound of $\{a, b\}$ and b_2 is minimal upper bound of $\{b_1, c\}$, then there is a minimal upper bound b_3 of $\{b, c\}$ such that b_2 is a minimal upper bound of $\{a, b_3\}$. This interpretation clarifies the meanings for the present notation. When this is followed, the meaning of the following proposition is unambigious.

Proposition 2.2 *The following identities are true in a quasi-lattice* $(P, \leq)$. *(A1):* $a \vee a = a$; *(A2):* $a \wedge a = a$; *(A3):* $a \vee b = b \vee a$; *(A4):*$a \wedge b = b \wedge a$; *(A5):* $a \vee (a \wedge b) = a = (a \wedge b) \vee a$; *(A6):*$a \wedge (a \vee b) = a = (a \vee b) \wedge a$; $\forall a, b \in P$.

Proof: Let us verify $a \vee (a \wedge b) = a$. Let a_1 be a maximal lower bound of $\{a, b\}$, and a_2 be a minimal upper bound of $\{a_1, a\}$. Then $a_2 = a$ because $a_1 \leq a$. Other relations can also be verified in this way.

If $(P, \leq)$ is an associative quasi-lattice, then it further has the properties: (A7) : $a \vee (b \vee c) = (a \vee b) \vee c$ and (A8) : $a \wedge (b \wedge c) = (a \wedge b) \wedge c$; $\forall a, b, c \in P$. It is known that the relations (A1) to (A8) characterize a lattice, when $a \vee b$ and $a \wedge b$ are unique elements (see Theorem 1 in Section 1 in Chapter 1 in [2]). It is to be proved that an associative quasi-lattice should be a lattice. For this purpose, let us introduce some changes in applications of the notations $\vee$ and $\wedge$. For a given poset $(P, \leq)$, $A \subseteq P$ and $B \subseteq P$, let $A \vee B$ (respectively, $A \wedge B$) denote the collection of all elements of the form $a \vee b$ (respectively, $a \wedge b$) with $a \in A$ and $b \in B$. So, for example, the relation $a \wedge (a \vee b) = a = (a \vee b) \wedge a$ will mean $\{a\} \wedge (\{a\} \vee \{b\}) = \{a\} = (\{a\} \vee \{b\}) \wedge \{a\}$. Thus a poset $(P, \leq)$ is a quasi-lattice if and only if $\{a\} \vee \{b\}$ and $\{a\} \wedge \{b\}$ are non empty subsets of P, for any $a, b \in P$. It is a lattice if and only if $\{a\} \vee \{b\}$ and $\{a\} \wedge \{b\}$ are singleton subsets of P, for any $a, b \in P$.

Theorem 2.3 *A quasi-lattice $(P, \leq)$ is associative if and only if it is a lattice.*

Proof: Suppose $(P, \leq)$ is an associative quasi-lattice. Let $x, y \in P$ and $a, b \in \{x\}\wedge\{y\}$.Then $a \leq y$, $a \leq x$ and $(\{x\}\wedge\{y\})\wedge\{a\} = \{x\}\wedge(\{y\}\wedge \{a\}) = \{x\}\wedge\{a\} = \{a\}$, when $(\{x\}\wedge\{y\})\wedge\{a\} \supseteq \{a, b\}\wedge\{a\} = \{a\}\cup(\{b\}\wedge\{a\})$.Thus $\{a\} \wedge \{b\} = \{a\}$ so that $a \leq b$. Similarly $b \leq a$ so that $a = b$.Thus $\{x\} \wedge \{y\}$ contains a unique element. Dually, $\{x\} \vee \{y\}$ contains a unique element. This proves that $(P, \leq)$ is a lattice.

3 Modular quasi-lattices

Definition 3.1 *A quasi-lattice $(P, \leq)$ is said to be modular if $\{x\} \vee (\{y\} \wedge \{z\}) = (\{x\} \vee \{y\}) \wedge \{z\}$ whenever $x, y, z \in P$ and $x \leq z$.*

Theorem 3.2 *A modular quasi-lattice $(P, \leq)$ is a lattice.*

Proof: Fix x, y in the given modular lattice $(P, \leq)$. Let $a, b \in \{x\} \wedge \{y\}$. Then $a \leq x, a \leq y, b \leq x$, and $b \leq y$. So, $(\{x\} \wedge \{y\}) \vee \{a\} = \{a\} \vee (\{y\} \wedge \{x\}) = (\{a\}\vee\{y\})\wedge\{x\} = \{y\}\wedge\{x\} = \{x\}\wedge\{y\}$, when $\{a, b\} \subseteq \{x\}\wedge\{y\}$. So $\{a, b\}\vee\{a\} \subseteq \{x\}\wedge\{y\}$ and hence $\{a, a\vee b\} \subseteq \{x\}\wedge\{y\}$.Thus $a\vee b \in \{x\}\wedge\{y\}$, when $a \vee b \geq a, a \vee b \geq b$, $a \in \{x\} \wedge \{y\}$ and $b \in \{x\} \wedge \{y\}$. So,the maximality of a and b implies that $a = a \vee b = b$. In particular, $\{x\} \wedge \{y\}$ contains atmost one point. Dually $\{x\} \vee \{y\}$ contains atmost one point. This proves the theorem.

Associative quasi-lattices are lattices and modular quasi-lattices are lattices. So it is difficult to derive new results for quasi-lattices, because quasi-lattices with additional fundamental properties become lattices. However, one can derive fundamental results for ideals.

Definition 3.3 *A subset I ($\mathscr{F}$, respectively) of a quasi-lattice $(P, \leq)$ is called an ideal (a filter, respectively), if*

(i) $a, b \in I \Rightarrow \{a\} \vee \{b\} \subseteq I$

((i) $a, b \in \mathscr{F} \Rightarrow \{a\} \wedge \{b\} \subseteq \mathscr{F}$*, respectively) and*

(ii) $a \in I, b \in P, b \leq a \Rightarrow b \in I$

((ii) $a \in \mathscr{F}, b \in P, b \geq a \Rightarrow b \in \mathscr{F}$*, respectively).*

An arbitrary intersection of ideals (filters) in a quasi-lattice is an ideal (a filter). The intersection of a filter and an ideal is sub quasi-lattice. Here, a sub quasi-lattice $(Q, \leq)$ of a quasi-lattice $(P, \leq)$ means that $\{x\} \vee \{y\} \subseteq Q$, and $\{x\} \wedge \{y\} \subseteq Q$, whenever $x, y \in Q$. The intersection of a filter with an ideal is a convex subset in view of the following (usual) definition.

Definition 3.4 *A subset C of a quasi-lattice $(P, \leq)$ is said to be convex, if $a \in C$, whenever $x, y \in C$, $a \in P$ and $x \leq a \leq y$.*

Notation 3.5 *To each $A \subseteq P$, a quasi-lattice, let $(A]$ and $[A)$ denote the smallest ideal and the smallest filter, respectively, containing A. They exist in view of the previous remark.*

Proposition 3.6 *Let $(P, \leq)$ be a quasi-lattice. Let $I(P)$ (respectively, $F(P)$) be the collection of all ideals (respectively, filters) of $(P, \leq)$. Then $I(P)$ (respectively, $F(P)$) is a complete lattice under the inclusion relation (respectively, inverse inclusion relation).*

Proof: Let $(I_\lambda)_{\lambda \in A}$ be a collection of ideals in P, Then $\cap\{I_\lambda : \lambda \in A\}$ and $(\cup\{I_\lambda : \lambda \in A\}]$ are ideals which are the greatest lower bound and the least upper bound of the given collection. A similar argument is applicable for filters.

4 Congruence relations

Ideals are associated with inverse image of a least element for a lattice homomorphism. A lattice homomorphism is associated with a congruence. Let us first define a congruence relation for a quasi-lattice.

Definition 4.1 *Let $(P, \leq)$ be a quasi-order lattice. An equivalence relation θ on P is denoted by $x \equiv y (mod\ \theta)$ when x and y are related in P by θ. Moreover, for subsets A,B of P, the identity $A \equiv B$ (mod θ) means the following:*

(i) to each $a \in A$, there is a $b \in B$ such that $a \equiv b$ (mod θ), and

(ii) to each $b \in B$, there is an $a \in B$ such that $a \equiv b$ (mod θ).

The equivalence relation θ on P is called a congruence relation, if $\{x_1\} \wedge \{y_1\} \equiv \{x_2\} \wedge \{y_2\} (mod\, \theta)$, and $\{x_1\} \vee \{y_1\} \equiv \{x_2\} \vee \{y_2\} (mod\, \theta)$, whenever $x_1 \equiv x_2 (mod\ \theta)$ and $y_1 \equiv y_2 (mod\ \theta)$ in P, and if $\{x\} \wedge \{y\} \subseteq [z]$, when $z \in \{x\} \wedge \{y\}$ and $\{x\} \vee \{y\} \subseteq [z]$, when $z \in \{x\} \vee \{y\}$, for x,y,z in P, when $[z]$ refers to the equivalence class containing z, determined by θ.

It is known that the collection of all partitions is a complete lattice under the "refinement" relation. The collection of all congruences on a lattice is a (complete) sublattice of the lattice of all partitions. In the same way(see the proof of theorem 3.9 in [1]), one can verify that the collection of all congruences on a quasi-lattice is a complete lattice and a sublattice of the lattice of all partitions.

Lemma 4.2 *Let $(P, \leq)$ be a quasi-lattice, and θ be a congruence relation on P. If $u \equiv v\ (mod\ \theta)$, $a \in \{u\} \wedge \{v\}$, $b \in \{u\} \vee \{v\}$, and if $a \leq x \leq b$, then $u \equiv x\ (mod\ \theta)$.*

Proof: Under the assumptions, we have $\{x\} = \{x\} \vee \{a\} \equiv \{x\} \vee (\{u\} \wedge \{v\}) \equiv \{x\} \vee (\{u\} \wedge \{u\}) \equiv (\{x\} \vee \{u\}) (\text{mod } \theta)$. Dually, we have $\{x\} = \{x\} \wedge \{b\} \equiv \{x\} \wedge (\{u\} \vee \{v\}) \equiv \{x\} \wedge (\{u\} \vee \{u\}) \equiv \{x\} \wedge \{u\} (\text{mod } \theta)$. So, we have $\{u\} = \{u\} \wedge (\{u\} \vee \{x\}) = \{u\} \wedge \{x\} \equiv \{x\} (\text{mod } \theta)$. This proves the lemma.

Definition 4.3 *Let $T : P_1 \to P_2$ be a mapping from a quasi-lattice P_1 into a quasi-lattice P_2. It is said to be a q-lattice homomorphism, if $T(\{x\} \vee \{y\}) = \{T(x)\} \vee \{T(y)\}$ and $T(\{x\} \wedge \{y\}) = \{T(x)\} \wedge \{T(y)\}, \forall x, y \in P$*

Definition 4.4 *Let θ be an equivalence relation on a quasi-lattice $(P, \leq)$. Let $[x]$ denote the equivalence class containing x. Let us say that θ satisfies the condition (*) if the following are true in P :*

(i) If $[x] \neq [y]$, $x \leq z$ and $y \leq z$, then there are elements $a \in [x]$ and $b \in [y]$ and there is an element $d \in \{a\} \vee \{b\}$ such that $d \leq z$.

(ii) If $[x] \neq [y]$, $x \geq z$ and $y \geq z$, then there are elements $a \in [x]$ and $b \in [y]$ and there is an element $d \in \{a\} \wedge \{b\}$ such that $d \geq z$.

Let us now state a fundamental theorem of homomorphism.

Theorem 4.5 *Let $(P, \leq)$ be a quasi-lattice. Let θ be a congruence relation on P that satisfies (*) of definition 4.4. Let P/θ be the collection of all equivalence classes. Let $[x]$ denote the equivalence class containing x. Then P/θ is a lattice in which we have $[x] \wedge [y] = [x \wedge y]$ and $[x] \vee [y] = [x \vee y]$, for any elements $x \wedge y$ and $x \vee y$ in $\{x\} \wedge \{y\}$ and $\{x\} \vee \{y\}$, respectively. Also, the quotient mapping $\pi : P \to P/\theta$ defined by $\pi(x) = [x]$, $x \in P$, is a surjective q-lattice homomorphism. On the other hand if $T : P \to L$ is a surjective q-lattice homomorphism from a quasi-lattice P onto a lattice L, then $\{T^{-1}(a) : a \in L\}$ defines a partition that leads to a congruence relation satisfying (*) of definition 4.4.*

Proof:
First Part: Define $[x] \leq [y]$ if and only if $a \leq b$ for some $a \in [x]$ and some $b \in [y]$. Suppose $a_1 \in [x]$ and $b_1 \in [y]$ such that $a_1 \leq b_1$. If $a_2 \in [x]$, then $a_1 \equiv a_2 \pmod{\theta}$, $a_2 \leq b_1 \vee a_2$ (for any element of this type) and $\{b_1\} \vee \{a_2\} \equiv \{b_1\} \vee \{a_1\} \equiv \{b_1\} \pmod{\theta}$. If $b_2 \in [y]$, then $b_1 \equiv b_2 \pmod{\theta}$, $a_1 \wedge b_2 \leq b_2$, and $\{a_1\} \wedge \{b_2\} \equiv \{a_1\} \wedge \{b_1\} \equiv \{a_1\} \pmod{\theta}$. Thus, if $[x] \leq [y]$, then for any $a_1 \in [x]$, there is a $b_1 \in [y]$ such that $a_1 \leq b_1$ and for any $b_2 \in [y]$ there is an $a_2 \in [x]$ such that $a_2 \leq b_2$. Now let us verify that this relation in P/θ is a partial order relation. Since $x \leq x$, we have $[x] \leq [x], \forall x \in P$. To prove anti-symmetricity, assume that $[x] \leq [y]$ and $[y] \leq [x]$ for two elements $x, y \in P$. Then there is an element $y_1 \in [y]$ such that $x \leq y_1$; and there is an element $x_1 \in [x]$ such that $y_1 \leq x_1$. Thus $x \leq y_1 \leq x_1$ and $x_1 \equiv x \pmod{\theta}$. By the previous lemma 4.2 it is concluded that $y_1 \equiv x \pmod{\theta}$. This proves that $\leq$ is anti-symmetric in P/θ. To prove transitivity, assume that $[x] \leq [y]$ and $[y] \leq [z]$ for some $x, y, z \in P$. Then there is an element $y_1 \in [y]$ and there is an element $z_1 \in [z]$ satisfying $x \leq y_1 \leq z_1$ so that $x \leq z_1$, So $(P/\theta, \leq)$ is a poset. To prove that P/θ is a lattice, consider an element $a \in \{x\} \wedge \{y\}$, for some fixed elements x, y. Then $a \leq x$ and $a \leq y$. So $[a] \leq [x]$ and $[a] \leq [y]$. Suppose $[b] \leq [x]$ and $[b] \leq [y]$ for some element b of P, and assume that $[a] \leq [b]$. Then there is an element $b_1 \in [b]$ such that $a \leq b_1$. There are elements $c_1 \in [x]$ and $c_2 \in [y]$ such that $b_1 \leq c_1$ and $b_1 \leq c_2$. By the condition (*) satisfied, there are elements $a_1 \in [c_1]$ and $a_2 \in [c_2]$ and there is an element $d \in \{a_1\} \wedge \{a_2\}$ such that $b_1 \leq d$. Since $\{a\} \equiv \{x\} \wedge \{y\} \equiv \{c_1\} \wedge \{c_2\} \equiv \{a_1\} \wedge \{a_2\} \equiv \{d\} \pmod{\theta}$, we have the relation $[b] \leq [a]$. Thus $[a] = [b]$. This proves that $[x] \wedge [y] = [x \wedge y]$ for any element $x \wedge y, \forall x, y \in P$. Dually, one can prove that $[x] \vee [y] = [x \vee y]$, for any element $x \vee y, \forall x, y \in P$. So, P/θ is a lattice. Other sub divisions of the

first part are trivial.

Second Part: Let θ be the equivalence relation induced by the partition $\{T^{-1}(a) : a \in L\}$. The condition (*) of definition 4.4 has to be checked to complete the proof as the other sub divisions are trivial. If $T^{-1}(a) \neq T^{-1}(b)$, $a_1 \in T^{-1}(a), b_1 \in T^{-1}(b), d_1 \geq a_1, d_1 \geq b_1$ and $T(d_1) = d$, then $d \geq a \vee b, \{a_1\} \vee \{b_1\} \subseteq T^{-1}(a \vee b)$, and $[a_1 \vee b_1] \leq [d_1]$ (in view of the order relation introduced in first part) so that there are $a_2 \in T^{-1}(a), b_2 \in T^{-1}(b)$ such that $d_1 \geq a_2 \vee b_2 \equiv a_1 \vee b_1 (\text{mod}\, \theta)$. Similarly, if $T^{-1}(a) \neq T^{-1}(b)$, $a_1 \in T^{-1}(a), b_1 \in T^{-1}(b)$, $d_1 \leq a_1$ and $d_1 \leq b_1$ then there are $a_2 \in T^{-1}(a), b_2 \in T^{-1}(b)$, such that $d_1 \leq a_2 \wedge b_2 \equiv a_1 \wedge b_1$ (mod θ). This completes the proof of the theorem.

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