Low and High Vertices in Edge Critical Graphs

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Abstract

If a graph G is k-edge critical, then its maximum degree is at most k-1. Motivated from this result, we define a vertex of a k-edge critical graph as a high vertex, if its degree is k-1 and as a low vertex, otherwise. In this Paper, we investigate the properties of the low and high vertices and the subgraphs induced by the sets of low or high vertices of a k-edge critical graph.

I. INTRODUCTION

The chromatic number $\chi(G)$ of a graph G. is one of the most popular graphical invariants. Dirac [1, 2] introduced the concept of k-critical graphs as the graphs, which are k-chromatic and are inclusion minimal with respect to this property. It is well known that if G is k-critical, then minimum degree of G is at least k —1. Motivated by this observation, Gallai [3, 4], introduced the concept of high and low vertices. A vertex of a k-critical graph is said to be a low or high vertex, according as its degree is k-1 or not. The subgraphs induced by the set of high vertices and the set of low vertices have been studied extensively by many people like Gallai [3, 4], Stiebitz [7], Sachs [5] and Sachs and Stiebitz [6]. A survey of the results about k-critical graphs having low vertices is given Sachs and Stiebitz [6]. These concepts play a significant role in characterizing and constructing edge-critical graphs.

The Pseudo achromatic number of a graph G is defined as the maximum number of colors that can be assigned to the vertices of G such that for any two distinct colors, there must exist an edge whose end vertices have those pair of colors and is denoted by $\psi_s(G)$. A graph G is called k-edge critical graph, if $\psi_s(G-e) < \psi_s(G)$, for every edge, e, of G. It was studies extensively by Suresh Kumar [8, 9, 10, 11].

If a graph G is k-edge critical, then its maximum degree is at most k-1. Motivated from this result, we define a vertex of a k-edge critical graph as a high vertex, if its degree is k-1 and as a low vertex, otherwise. In this paper, we introduce the concepts of low and high vertices in edge critical graphs with respect to pseudo-achromatic number and study their properties.

II. MAIN RESULTS

Proposition 2.1. If G is k-edge critical, then its maximum degree is at most k-1.

Proof. Let G be a k-edge critical graph. If there is a vertex v of G, of degree greater than k-1, then either there exists a pair of colors such that the end vertices of at least two edges incident with v receive that pair of colors or a neighbour of v has the same color as that f v and hence G is not k-edge critical. Thus maximum degree of G is at most k-1.

The above Proposition motivates us to define and study the following definition similar to the works done by Gallai [49,50] for critical graphs with respect to chromatic number.

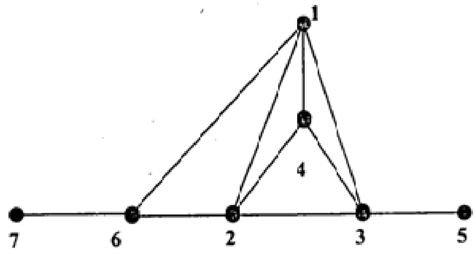
Definition 2.1. A vertex v of a k-edge critical graph G is called a high vertex if its degree is k-1. Otherwise it is called a low vertex. The subgraph induced by the set of all high (low) vertices of G is called the high (low) vertex subgraph of G. Throughout this paper, G stands for a k-edge critical graph without isolated vertices, whose vertex set and edge set are V(G) and E(G) respectively.

Theorem 2.2. A vertex v of G is a high vertex of G if and only if in any k-pseudo complete coloring of G, the color of v is different from that of any other vertex of G.

Proof. Suppose there is a k-pseudo-complete coloring of G, in which a high vertex v and another vertex u receive the same color, say c. Let d be the color assigned to a neighbour of u. Now, there exists two edges whose end vertices receive the pair of colors (c,d) and hence G is not k-edge critical, a contradiction. Conversely suppose that in any k-pseudo-complete coloring of G, the color of v is different from that of any other vertex of G. Then v is adjacent to at least k-i vertices, since for any two colors, there should be a pair of adjacent vertices with those colors. Thus v is a high vertex of G.

Corollary 2.3 The high vertex subgraph of G is complete.

However, the high vertex subgraph of G need not be a clique of G. The graph given below is a 5-edge-critical graph whose high vertex subgraph is induced by the vertices,{1,2,3}, which is not a clique of G.



Corollary 2.4. G has no low vertices if and only of G = K_k.
Corollary 2.5. G has exactly two (three) low vertices if and only if G can be obtained from the complete graph on k vertices by 2-splitting (3-splitting) one of its vertices.
Corollary 2.6. There is no edge critical graph with exactly one low vertex.
Corollary 2.7. The low vertex subgraph of G is empty if and only if G can be obtained from the complete graph on k vertices, by n -splitting one of its vertices for some n >1.

It follows from the Corollary 2.3 that a k-edge critical graph can have at most k high vertices. The following theorem gives bounds for the number of vertices of a k-edge critical graph.

Let ⊗ denote the graph operation defined as $G\otimes H$ denote the graph obtained from G by joining each of its vertex to every vertex of H. Thus, $K_m\otimes\overline{K}_{k-m}$ denote the graph obtained from the complete subgraph of order m when each of its vertex is adjacent to exactly k-m new pair wise non-adjacent vertices.
Theorem 2.8. Let G be a k-edge critical graph with exactly rn high vertices. Then $2k-m\leq|V(G)|\leq 2\binom{k-m}{2}+m(k-m+1)$
Proof. It follows from Theorem 2.2 that the number of low vertices is at least twice the number of distinct colors represented among the low vertices in any k-pseudo-complete coloring of G. Hence |V(G)| ≥m + 2(k - m) = $2k-m$. Now, |V(G)|=2k-m if and only if G has 2(k-m) low vertices.

Also by Theorem 2.2., for each of the $k-m$ colors, which are not represented among the high vertices, there exist exactly two low vertices with that color with respect to any k-pseudo-complete coloring of G. Hence G can be obtained from the complete graph on k vertices by 2-splitting k-m of its vertices and conversely. Now, by Corollary 2.5, the high vertex subgraph of G is a complete subgraph of order m and each vertex is adjacent to exactly k-m low vertices. Further, for any two of these k-m colors there exists an edge in the low vertex subgraph whose end vertices receive those colors. Hence the low vertex subgraph has at least $\binom{k-m}{2}$ edges.

Hence G has maximum number of vertices if and only if G is the disjoint union of $\binom{k-m}{2}$ copies of K_2 and $K_m\otimes\overline{K}_{k-m}$. So, $|V(G)|=2\binom{k-m}{2}+m(k-m+1)$.

Further, it follows from the above proof that $|V(G)|=2k-m$ if and only if G can be obtained from the complete graph on k vertices by 2-splitting k-m of its vertices and $|V(G)|=2\binom{k-m}{2}+m(k-m+1)$if and only if G is the union of $\binom{k-m}{2}$ copies of K_2 and the graph $K_{m}\otimes\overline{K}_{k-m}$
Theorem 2.9. A vertex v of a k-edge critical graph G is a high vertex of G if and only if G-v is a (k-1)-edge critical graph (with possibly isolated vertices).
Proof. If G-v is (k- 1)-edge critical and G is k-edge critical, then the degree of in G is equal to $\binom{k}{2}-\binom{k-1}{2}=k-1$ and hence v is2 2a high vertex of G. Conversely, let v be a high vertex of G. Then G-v is (k-1)-pseudo-complete colorable. Also, $|E(G)|=|E(G-v)|-k-1=\binom{k-1}{2}$and hence G--v is (k-1)-edge critical.
Corollary 2.10. The low vertex subgraph of G is either an empty graph or an edge critical graph.
Proposition 2.11. If a high vertex v of G is replaced by a complete graphK_n such that every vertex of $K_n$is adjacent to every neighbour of v, then the resulting graph H is an edge critical graph whose high vertex subgraph contains K_n
Proof. By Theorem 2.2, in any k-pseudo-complete coloring of G, the color of v is different from that of any other vertex of G. Now, by assigning n new distinct colors to the vertices of$K_n$we get a (k + n-1)-pseudo-complete coloring of H. Clearly, H is (n+k-1)-critical and its high vertex subgraph contains K_n.

Corollary 2.12. Given any non-negative integer *n* and a k-edge critical graph G with *h* high vertices, there is an (n+ k)-edge critical graph H whose high vertex sub graph is K_{n+h} and the low vertex subgraph is the same as that of G.

Corollary 2.13. Given any non-negative integer n and a k-edge critical graph G without high vertices, there is an (n+ k)-edge critical graph H whose high vertex subgraph is K_n and the low vertex subgraph is G.

The following theorem shows the usefulness of the concept of low and high vertices in constructing new edge critical graphs. Let S, A and B be subsets of V(G). Let $\langle S\rangle$denote the subgraph induced by the vertices in S and let [A, B] denote the set of all edges with one end in A and the other end in B.

Theorem 2.14. Let G_i be a k_i -edge critical graph for each i =1,2...m where m>1 and k_i>1, for each i =1,2...m. Then, there existsa k-edge critical graph G where $k = k_1 + k_2 +...+ k_m$ such that V(G) can be partitioned into subsets $\{V_i\}_{i=1}^{m}$ with $\langle V_i\rangle \cong G_i$, for each i =1,2,...,m.

Proof. It is enough to prove the theorem for m = 2. Suppose G_1 has h high vertices, where h is a non-negative integer. Then, by Corollary. 2.12, there is a $(k_1 + k_2)$-edge critical graph H, whose high vertex subgraph is a $K_{k_2+h}$and the low vertex subgraph is the same as that of G_1.

Let $\{v_1, v_2, ..., v_{k_2+h}\}$ be the set of all high vertices of H and let $\{u_1, u_2 \dots\dots u_{p-h}\}$ where p = |V(G_1)| be the set of all low vertices of H. Now put S= $\{v_1, v_2, ..., v_{k_2}\}$. Clearly $\langle S\rangle = K_{k_2}$. So, G_2 can be obtained from $K_{k_2}$by a sequence of n-splitting operations on its vertices. Let G be the graph obtained from H, by performing a sequence of n-splitting operations on the vertices of S in such a way that the subgraph induced by S reduces to a graph which is isomorphic to G_2.Thus G satisfies the required conditions.

REFERENCES

[1] Dirac G.A., Some theorems on abstract graphs, Proc. London Math. Soc.,2, (3) (1952), 69-8 1.

[2] Dirac G.A., A property of 4-chromatic graphs and some remarks on critical graphs, J. London Math. Soc., 27, (1952), 85-92.

[3] Gallai T., Kritische Graphen I , Magyar Tud A/cad. Mat. Kutato. mt. K6 z1 ,8, (1963), 165-192.

[4] Gallai 1., Kritische Graphen II, Magyar Tud Akad. Mat. Kutato. mt. K6 z1 ,8, (1963), 373-395

[5] Sachs H., On colour critical graphs, Fundamentals of Computation theory, FCT '85. Cottbus, GDR, Sept. 1985. Lecture notes in Computer Science, Springer Verlag, (1985), 390-401.

[6] Sachs H. and M.Stiebitz, Construction of. colour-critical graphs with given major-vertex subgraph, Combinatorial Mathematics, Annals .of Discrete Math., 17, (1983), 581- 598.

[7] Stiebitz M., Beitrrrrge zur theorie der farbungs kritischen graphen, Dissertationsschrfl (B), Technische Hochschule Ilmenau (1985)

[8] SureshKumar J and D Satheesh E.N, Pseudo-Complete Color Critical Graphs, International Journal for Research in Applied Science & Engineering Technology (IJRASET) Volume 6, Issue I, January 2018.

[9] Suresh Kumar J, Degrees and Degree Sequences of PAN Critical Graphs, International Journal for Mathematics And its Applications (IJMAA), 6(1–B)(2018)

[10] Suresh Kumar J, Diameter and Traversability of PAN Critical Graphs International Journal of Mathematics Trends and Technology (IJMTT), Volume.52, Number 7, December 2017

[11] Suresh Kumar J, On Pseudo-Complete Colorings and Graphoidal colorings of a graph, Ph.D. Thesis, M.S. University, Thirunelveli, Tamilnadu, India, 2002.