Locally Critical Elements in Graphs

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Abstract

A vertex of a k-edge critical graph is called a high vertex, if its degree is k-i and is called a low vertex, if otherwise. In this Paper, we investigate the properties of the low and high vertices and the subgraph induced by the set of low vertices and the set of high vertices of a k-edge critical graph.

1. INTRODUCTION

The Pseudo achromatic number of a graph G is one of the most popular graphical invariants and introduced by Gupta [] and studied by Sampathkumar and Bhave [] using partition graphs and formally studied by Suresh Kumar [1] in his Doctoral thesis. The Pseudo achromatic number of a graph G is defined as the maximum number of colors that can be assigned to the vertices of G such that for any two distinct colors, there must exist an edge whose end vertices have those pair of colors and is denoted by $\psi_s(G)$. This parameter generalizes a well-known graph coloring parameter called the achromatic number of a graph G, where in addition to the above condition, it is also required that the adjacent vertices have distinct colors.

A graph critical with respect to the Pseudo achromatic number is called a critical graph. It is well studied by Suresh Kumar [1, 2,3,4]. In this Paper, we introduce three types of locally critical (with respect to pseudo-achromatic number) elements of graphs such as of critical vertices, critical edge and non-contractible edges and study their properties.

Let G be a graph. Let v be a vertex of G. Then $G - v$ denote the graph obtained by removing the vertex, v (and hence the edges incident at v also). If G is a graph and e is an edge of G, then $G - e$ denote the graph obtained by removing the edge, e.

Let G be a graph. Let $e = \{u, v\}$ be an edge of G. Then $G||e$ denote the graph obtained by contracting the edge, e (that is, removing both u and v and add a new vertex w which is adjacent to a vertex to which either u or v is adjacent).

II. CRITICAL VERTICES, CRITICAL EDGES AND NON-CONTRACTIBLE EDGES

Throughout this section, by criticality we mean the criticality with respect to the Pseudo achromatic number of a graph G.

Definition 2.1.

A vertex v of a graph G is called a *critical vertex*, if $\psi_s(G - v) < \psi_s(G)$.

An edge e of a graph G is called a *critical edge*, if $\psi_s(G - e) < \psi_s(G)$.

An edge e of a graph G is called a *non-contractible edge* of G, if$\psi_s(G||e) < \psi_s(G)$.

Theorem 2.2. The end vertices of a critical edge of a graph G are critical vertices of G.

Proof. Since G is edge-critical, for any edge e of G, $\psi_s(G - e) < \psi_s(G)$. Hence it follows that an edge, e = {u, v} is a critical edge of agraph G with $\psi_s(G) = k$ if and only if in any k-pseudo-complete coloring of G, there exists two colors {c, d} such that u and v are the only adjacent vertices, having the colors c and d respectively. Hence the end vertices of a critical edge are critical. ■

Converse of this theorem is not true. That is, a critical vertexof a graph G need not be the end vertexof some critical edgeof G. For example, in the graph $P_{11} + K_1$, the unique vertex of degree 11 is critical, but none of the edges incident at that vertex is critical.

Theorem 2.3. Any critical edge of a graph G is a non-contractible edge of G.

Proof. Since $G||e$ is obtained from G by contracting the edge, e={u, v} (that is, by removing u, v and by adding a new vertex w which is adjacent to vertices to which either u or v is adjacent), G-e can be obtained from G||e, by a 2-splitting operation on the new vertex. Hence any critical edge is non-contractible edge. ■

Again, converse of this theorem is not true. That is, a non-contractible edge of a graph G need not be critical edge of G. For example, in thegraph $P_{11} + K_1$, any edge incident to the unique vertex of degree 11 is non-contractible, but is not critical.

Theorem 2.4. The end vertices of a non-contractible edge of a graph G are critical vertices of G.

Proof. We observe that an edge $e = \{u, v\}$ is non-contractible if and only if in any k-pseudo-complete coloring of G, u and v have distinct colors, where $k = \psi_s(G)$. Hence, the end vertices of a non-contractible edge are critical.■

Again, converse of this theorem is not true. That is, a critical vertex of a graph G need not be an end vertex of some non-contractible edge of G. For example, every vertex of the cycle graph on 4 vertices is critical, but none of its edges is non-contractible.

The following proposition gives a condition for a critical vertex of a graph G to be an end vertex of some non-contractible edge of G. We observe that if a vertex v of G (which has p vertices) has degree p-1, then v is a critical vertex of G.

Proposition 2.5. Let G be a graph on p vertices and let v be a vertex of G with degree p-1. Then, any vertex u of G, $u \neq v$, is critical vertex of G if and only if the edge $e = \{u, v\}$ is a non-contractible edge of G.

Proof. Suppose that u is a critical vertex of a graph G with $u \neq v$, and if possible suppose that the edge, $e = \{u, v\}$ is not a non-contractible edge. Then, $\psi_s(G) - 1 \leq \psi_s(G - \{u, v\}) \leq \psi_s(G - v) = \psi_s(G) - 1$. Thus, $\psi_s(G - \{u, v\}) \leq \psi_s(G - v)$ so that u is not a critical vertex of G—v and hence u is not a critical vertex of G, which is a contradiction to the choice of u. Hence the edge $e = \{u, v\}$ must be a non-contractible edge. Converse follows from theorem 2.4. ■

Corollary 2.6. If G is, then $G + K_n$, is a $(k + n)$-con-critical graph.

Proof. Consider any k-pseudo-complete coloring of G. Then it can be extended to a (k+n)-pseudo-complete coloring of $G + K_n$, by assigning n new colors to the vertices of K_n. Let e be any edge of $G + K_n$. If e is an edge of G, then $\psi_s(G||e) < \psi_s(G)$ so that $\psi_s\big((G + K_n)||e\big) < \psi_s(G + K_n,)$If e is an edge joining a vertex of G with a vertex of K_n, then by previous proposition, e is a non-contractible edge. Thus, $\psi_s\big((G + K_n)||e\big) < \psi_s(G + K_n,)$.Hence, $G + K_n$, is a $(k + n)$-con-critical graph.■

Finally, we observe that if G is edge critical, G + K_1 need not be edge critical. But if G is k-vertex critical, then clearly, G + K_1 is (k+1)-vertex critical.

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