Fixed Point Theorems in Complete G - Metric Spaces

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Abstract

The main purpose of this paper is to prove some fixed point results for mappings satisfying various contractive conditions on Complete $G-$Metric spaces. We also prove the uniqueness of such fixed points as well as we showed these mappings are G-continuous on such fixed points.

Keywords: *Common Fixed Point, G-metric spaces, weakly compatible mapping,*

I. INTRODUCTION AND PRELIMINARIES

The study of metric fixed point theory has been researched extensively in the past decades, since fixed point theory plays a major role in mathematics and applied sciences, such as optimization, mathematical models, and economic theories.

Different mathematicians tried to generalize the usual notion of metric space(X,d) such as Gahler [3] and Dhage [1,2] to extend known metric space theorems in more general setting, but different authors proved that these attempts are invalid.

In 2005, Mustafa and Sims [4] introduced a new structure of generalized metric spaces which are called G-metric spaces as generalization of metric space (X,d) *to* develop and introduce a new fixed point theory for various mappings in this new structure.

Definition 1.1[4]. Let X be a non- empty set, and let G: X × X × X → R^+ be a function satisfying the following axioms:

($\mathbf{G_1}$) $G(x,y,z) = 0 \; if \; x = y = z,$

($\mathbf{G_2}$) $0 < G(x,x,y)$, for all $x,y \in X, with \; x \neq y,$

($\mathbf{G_3}$) $G(x,x,y) \leq G(x,y,z), for \; all \; x,y,z \in X, with \; z \neq y,$

($\mathbf{G_4}$) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables),

($\mathbf{G_5}$) $G(x,y,z) \leq G(x,a,a) + G(a,y,z), for \; all \; x,y,z,a \in X,$ (Rectangle inequality).

then the function G is called a generalized metric or more specifically a G-metric on X, and the pair (X,G) is called a G-metric space.

Example 1.2 Let R be the set of all real numbers. Define $G: R \times R \times R \to R^+$ by

$$G(x,y,z) = |x-y| + |y-z| + |z-x|, for \; all \; x,y,z \in X.$$

Then it is clear that (R,G) is a G-metric space.

Proposition 1.3[4]:- Let (X,G) be a G-metric space. Then for any x,y,z, and $a \in X$, it follows that

(1) $If \; G(x,y,z) = 0, then \; x=y=z,$
(2) $G(x,y,z) \leq G(x,x,y) + G(x,x,z),$
(3) $G(x,y,y) \leq 2G(y,x,x),$
(4) $G(x,y,z) \leq G(x,a,z) + G(a,y,z),$
(5) $G(x,y,z) \leq (\frac{2}{3})(G(x,y,a) + G(x,a,z) + G(a,y,z)),$
(6) $G(x,y,z) \leq (G(x,a,a) + G(y,a,a) + G(z,a,a)).$

Definition 1.4 Let (X,G) and (X',G') be G-metric spaces and let $f : (X,G) \to (X',G')$ be a function, then f is said to be G-continuous at a point a ∈ X if given $\epsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in X;\ G(a, x, y) < \delta \text{ implies } G'(f(a), f(x), f(y)) < \epsilon .$$

A function f is G −continuous on X if and only if it is G-continuous at all $a \in X$.

Definition 1.5[4]:- Let (X, G) be a G-metric space. Then for $x_0 \in X, r > 0$, *the G − ball* with centre x_0 and radius r is :

$$B_G(x_{0,} r) = \{y \in X: G(x_0, y, y) < r\} (1.2)$$

Proposition 1.6[4]:- Let (X, G) be a G-metric space. Then for any $x_0 \in X, r > 0$ one has

(1) if $G(x_0, x, y) < r$, then $x, y \in B_G(x_{0,} r)$,
(2) if $y \in B_G(x_{0,} r)$, then there exists a $\delta > 0$ such that $B_G(y, \delta) \subseteq B(x_{0,}, r)$.

Definition 1.7:- A G-metric space $(X, G)$is called symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$ and called Nonsymmetric if it is not Symmetric.

Example 1.8 Let (R, d) be the usual metric space. Define $G_s$and G_m by

$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z)$, and

$G_m(x, y, z) = max\{d(x, y), d(y, z), d(x, z)\}$

for all $x, y, z \in R$. Then (R, G_s) and (R, G_m) are symmetric G-metric spaces.

Example 1.9. Let $X = \{a, b, c\}$ and define $G : X \times X \times X \longrightarrow R^+$ by,

$G(x, y, z) = 0 \; if \; x = y = z$

$G(a, b, b) = G(b, a, a) = 22$

$G(a, c, c) = G(c, a, a) = 27$

$G(b, c, c) = G(c, b, b) = 30,$

$G(a, b, c) = 35$

extended by symmetry in the variables. It is easily verified that G *is* a symmetric G-metric, but $G \neq G_s$ or G_m for any underlying metric.

Proposition 1.10[4]:- Every G-metric space (X, G) will define a metric space (X,d_G) by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \forall x, y \in X \qquad (1.1)$$

If (X, G) is a symmetric G −metric space, then

$$d_G(x, y) = 2G(x, y, y), \forall x, y \in X \qquad (1.2)$$

However, if (X, G) is not symmetric, then it holds by the G −metric properties that

$$\frac{3}{2} G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y), \quad \forall x, y \in X (1.3)$$

and that in general these inequalities cannot be improved.

II. MAIN RESULT

Lemma 2.1 Let (X, G) be a G-metric space . If $\lim_{n\to\infty} a_n = 0$then the sequence $\{x_n\}$ is a Cauchy sequence.

Proof : Since $\lim_{n\to\infty} a_n = 0$, we have for every $\varepsilon > 0$, there exists $m \in N$ such that for every

$n > m, |a_n - 0| < \varepsilon$ i.e. $\delta_G(A_n) < \varepsilon$

Then for $l, m, k \geq n > m$, we have

$$G(x_l, x_m, x_l) \le sup\{G(x_i, x_j, x_p): x_i, x_j, x_p \in A_n\} = \boldsymbol{a_n} < \varepsilon$$

Therefore $\{x_n\}$ is a Cauchy sequence in X.

Theorem 4.3.2 Let S,R,T,U be self-mapping of a complete G-metric space (X, G) satisfying

(i) $SR \subseteq TU$ and $TU(X)$ is a closed subset of X,

(ii) The pair (SR, TU) is weakly compatible,

(iii) $\int_0^{G(SRx,SRy,SRz)} \delta(t)dt \le \left(\int_0^{\phi(G(TUx,TUy,TUz))} \delta(t)dt\right)$ for every $x, y, z \in X$, where $\phi: [0, \infty) \to [0, \infty)$ is a non-decreasing continuous function with $\phi(t) < t$ for every $t > 0$ and $\delta(t)$ is a Lebesgue integrable function which is summable nonnegative such that

$$\int_0^{\varepsilon} \delta(t)dt > 0, \forall \varepsilon > 0$$

(iv) $(S, R), (T, U)$ are commutative then S, R, T, U have a common fixed point in X.

Proof : Let $x_0$be an arbitrary point in X. By (i) we can choose a point x_1 in X such that

$$y_0 = SRx_0 = TUx_1$$

And $y_1 = SRx_1 = TUx_2$. In general ,∃ a sequence $\{y_n\}$ such that

$y_n = SRx_n = TUx_{n+1}$ for $n = 0,1,2,3 \ldots.$

We prove that the sequence $\{y_n\}$ is a Cauchy sequence.

Let $A_n = \{y_n, y_{n+1}, y_{n+2} \ldots.\}$ And $a_n = \delta_G(A_n), n \in N$.

Then we know that $\lim_{n\to\infty} a_n = a$ for some $a \ge 0$.

Taking $x = x_{n+k}, y = y_{m+k}$and $z = z_{l+k}$ in (iii) for $k \ge 1$ and $m, n, l \ge 0$, we have

$$\int_0^{G(y_{n+k},y_{m+k},y_{l+k})} \delta(t)dt = \int_0^{G(SRx_{n+k},SRx_{m+k},SRx_{l+k})} \delta(t)dt$$

$$\le \left(\int_0^{\phi(G(TUx_{n+k},TUx_{m+k},TUx_{l+k}))} \delta(t)dt\right)$$

$$= \left(\int_0^{\phi(G(y_{n+k-1},y_{m+k-1},y_{l+k-1}))} \delta(t)dt\right) \qquad (1)$$

Now we claim $G(y_{n+k-1}, y_{m+k-1}, y_{l+k-1}) \le a_{k-1}$ for every $n, m, l \ge 0$.

Since $A_{k-1} = \{y_{k-1}, y_k, y_{k+1} \ldots.\}$, $a_{k-1} = sup\{G(a, b, c): a, b, c \in A_{k-1}\}$

Also $y_{n+k-1}, y_{m+k-1}, y_{l+k-1} \subseteq A_{k-1}$ implies $G(y_{n+k-1}, y_{m+k-1}, y_{l+k-1}) \le a_{k-1}$

Also ϕ is increasing in ,

From (1.1) we get $\sup_{m,n,j\ge 0} G(y_{n+k-1}, y_{m+k-1}, y_{l+k-1}) \le \phi(a_{k-1})$

Therefore we have $a_{k-1} \le \phi(a_{k-1})$, letting $k \to \infty$, we get $a \le\le \phi(a)$. If $a \ne 0$, then $a \le \phi(a) \le a$, which is a contradiction. Thus $a = 0$. Hence $\lim_{k\to\infty} a_n = 0$.

Thus by lemma ,$\{y_n\}$ is a Cauchy sequence in X, there exists $y_1 \in X$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} SRx_n = \lim_{n\to\infty} TUx_{n+1} = y_1.$$

Also $TU(X)$ is closed, there exists $z \in X$ such that $TUz = y_1$. Now we show that $SRz = y_1$. For this set x_n, x_n, z replacing x, y, z respectively in equation (iii) , we get

$$\int_0^{G(SRx_n,SRx_n,SRz)} \delta(t)dt \leq \left(\int_0^{\phi(G(TUx_n,TUx_n,TUz))} \delta(t)dt \right)$$

Taking $n \to \infty$, we get

$$\int_0^{G(y_1,y_1,SRz)} \delta(t)dt \leq \left(\int_0^{\phi(G(y_1,y_1,y_1))} \delta(t)dt \right) = 0$$

Implies $SRz = y_1$. Since the pair (SR, TU) is weakly compatible , hence we get$(SR)(TU) = (TU)(SR)z$. Thus $SRy_1 = TUy_1$.

Now we prove that $SRy_1 = y_1$. If we substitute x, y, z in (iii) by x_n, x_n, y_1 respectively

$$\int_0^{G(SRx_n,SRx_n,SRy_1)} \delta(t)dt \leq \left(\int_0^{\phi(G(TUx_n,TUx_n,TUy_1))} \delta(t)dt \right)$$

Taking $n \to \infty$, we get

$$\int_0^{G(y_1,y_1,SRy_1)} \delta(t)dt \leq \left(\int_0^{\phi(G(y_1,y_1,,TUy_1))} \delta(t)dt \right) = \left(\int_0^{\phi(G(y_1,y_1,,SRy_1))} \delta(t)dt \right)$$

If $SRy_1 \neq y_1$, then$\int_0^{G(y_1,y_1,SRy_1)} \delta(t)dt \leq \int_0^{G(y_1,y_1,SRy_1)} \delta(t)dt$ is a contradiction.

Therefore $SRy_1 = TUy_1 = y_1$.

For uniqueness let y_1 and y_2 be fixed points of SR, TU.

Taking $x = y = y_1$ and $z = y_2$ in (iii) we have

$$\int_0^{G(y_1,y_1,y_2)} \delta(t)dt = \int_0^{G(SRy_1 SRy_1,SRy_2)} \delta(t)dt \leq \left(, \int_0^{\phi(G(TUy_1,TUy_1,TUy_2))} \delta(t)dt \right)$$

$$= \left(\int_0^{\phi(G(y_1,y_1,y_2))} \delta(t)dt \right) < \left(\int_0^{(G(y_1,y_1,y_2))} \delta(t)dt \right)$$

a contradiction. Thus we have $y_1 = y_2$.

Now by (iv)$S, R), (T, U)$ are mutually commutative pair of mapping.

Consider $Sy_1 = S(SRy_1) = S(RSy_1) = SR(Sy_1)$, implies Sy_1 is the unique point of SR but y_1 is the unique fixed point of SR, hence $Sy_1 = y_1$.

Also $Ry_1 = R(SRy_1) = (RS)(Ry_1) = SR(Ry_1)$, implies $Ry_1$is the fixed point of SR, buty_1

is the unique fixed point of SR. Hence $Ry_1 = y_1$.

Thus $Sy_1 = Ry_1 = y_1$. In the same way we have $Ty_1 = Uy_1 = y_1$.

Hence the result.

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