On Study Generalized $\mathcal{BP}$ – Recurrent Finsler Space

Alaa A. Abdallah[1*], A.A. Navlekar[2] and Kirtiwant P. Ghadle[1]
[1]Department of Mathematics, Dr.Babasaheb Ambedkar Marathwada University, Aurangabad, 431004, India.
[2]Department of Mathematics Pratishitan Mahavidyalaya, Paithan (M.S.) India.

Abstract

In this paper, we introduced a Finsler space whichCartan's second curvature tensor P^i_{jkh} satisfies the generalized recurrence property in sense of Berwaldi.e. characterized by the following condition

$$\mathcal{B}_l P^i_{jkh} = \lambda_l P^i_{jkh} + \mu_l(\delta^i_j g_{kh} - \delta^i_k g_{jh}), \quad P^i_{jkh} \neq 0,$$

where $\mathcal{B}_l$ is Berwald's covariant differential operator with respect to x^l, λ_l and μ_l are known as recurrence vectors, such space is called a generalized $\mathcal{BP}$ −recurrentspace.

The aim of this paper study the properties of the above space by Berwald covariant derivative of first order. We obtain the associate curvature tensor P_{ijkh}, the torsion tensor P^i_{kh} and the associative torsion tensor $P_{jkh}$are non− vanishing. Also we obtain P − Ricci tensor P_{jk}, the curvature vector P_k and the curvature scalar P behave as recurrent. Also we obtain the tensor $(S^i_{jkh|r}\, y^r)$, the tensor $(C^i_{kh|j} + C^i_{sj} P^s_{kh} - j/k)$ is generalized recurrent. We obtain certain identities satisfy in the generalized $\mathcal{BP}$ − recurrent space.We discuss the projection of the generalized recurrence property on indicatrix with respect to Berwald's connection for the tensors whose behave as recurrent.

Keywords - *Finsler Space, Generalized $\mathcal{BP}$ – Recurrent Space, curvature tensor, associate curvature tensor, torsion tensor, Ricci Tensor, curvature vector, curvature scalar, projection on indicatrix.*

I. INTRODUCTION

Verma [11] discussed the recurrence property of the tensors W^i_{jkh},$H^i_{jkh}$and W^i_h in the sense of Berwald, Qasem[5] introduced and studied the curvature tensor U^i_{jkh} which satisfies the recurrence property, Pandeyat al. [10] introduced and studied a generalized H – recurrent Finsler space, Mohammed [1] introduced and studied P^h − Recurrent Space,Awed [2] introduced and studied generalized P^h − Recurrent Space Qasem and Baleedi [7] introduced and studied on a generalized $\mathcal{BK}$ −recurrent space, Qasem and Abdallah [6] introduced and studied a generalized $\mathcal{BR}$ − recurrent space,Zafar and Musavvir [4] introduced and studied on some properties of W – curvature tensor.Pandey [9] introduced and studied some problems in Finsler spaces.
Dikshit [12] studied the projection of some tensor onindicatrix with respect to Berwald's connection. Qasem [5] studied the projection of the curvature tensorR^i_{jkh} on indicatrix with respect to Cartan's connection. Hanballa [3] studied the projection on indicatrixin the P^h − birecurrent space, P^h −generalized birecurrent space with respect to Cartan's connection.

Let F_n be an n – dimensional Finsler space equipped with the metric function *F(x,y)* satisfying the request conditions[8] .
The vector y_i is defined by

(1.1) $\quad y_i = g_{ij}(x,y)y^j$.

The two sets of quantities g_{ij} and its associative g^{ij}, which are components of a metric tensor connected by

(1.2) $\quad g_{ij}\, g^{ik} = \delta^k_j = \begin{cases} 1 & if\ j = k, \\ 0 & if\ j \neq k. \end{cases}$

In view of (1.1) and (1.2), we have

(1.3) $\quad$ a) $\delta^i_k y^k = y^i$ and $\quad$ b) $\delta^i_j g_{ir} = g_{jr}$.

the unit vector l^i in the direction of y^i is given by

(1.4) $\quad l^i := \frac{y^i}{F}$

In particular the metric tensor g_{ij} and the associate metric tensor g^{ij} are covariant constant with respect to h −covariant derivative, i.e.

(1.5) $\quad a)\ g_{ij|k} = 0 \quad$ and $\quad b)\ g^{ij}_{\ \ |k} = 0$.

The h − covariant derivative of the vector $y^i$and $y_i$vanish identically, i. e.

(1.6) $\quad a)\ y^i_{\ |k} = 0, \quad$ and $\quad b)\ y_{i|k} = 0$

Berwald covariant derivative $\mathcal{B}_k T^i_j$ of an arbitrary tensor field T^i_j with respect to x^k is given by

$$\mathcal{B}_k T^i_j := \partial_k T^i_j - (\dot{\partial}_r T^i_j) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}.$$

Berwald covariant derivative of the vector y^i vanish identically, i.e.

(1.7) $\mathcal{B}_k y^i = 0.$

But, in general, Berwald covariant derivative of the metric tensor g_{ij} does not vanish and given by

(1.8) $\mathcal{B}_k g_{ij} = -2y^h \mathcal{B}_h C_{ijk}.$

The $h-$ curvature tensor (Cartan's third curvature tensor) is defined by [8]

$$R^i_{jkh} = \partial_h \Gamma^{*i}_{jk} + (\partial_l \Gamma^{*i}_{jk}) G^l_h + C^i_{jm} (\partial_k G^m_h - G^m_{kl} G^l_h) + \Gamma^{*i}_{mk} \Gamma^{*m}_{jh} - k/h^{\ *}.$$

This tensor satisfies the following relations

(1.9) $R^i_{jkh} y^j = H^i_{kh}$

The associate curvature tensor R_{rjkh} satisfies

(1.10) $R_{ijkh} = g_{ir} R^r_{ikh}.$

Also, the curvature tensor R^i_{jkh} and its associative tensor R_{ijhk} satisfies the following identities known as Bianchi identity [8]

(1.11) $a)\ R^r_{ijk|h} + R^r_{ihj|k} + R^r_{ikh|j} + y^m (R^s_{mkh} P^r_{ijs} + R^s_{mjk} P^r_{ihs} + R^s_{mhj} P^r_{iks}) = 0$

The $hv-$curvature tensor P^i_{jkh} (Cartan's second curvature tensor) is defined by [8]

(1.12) $P^i_{jkh} := \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jr} P^r_{kh} - C^i_{jh|k}$

or equivalent by

(1.13) $P^i_{jkh} := \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jr} C^r_{kh|s} y^s - C^i_{jh|k}$

The tensor P^i_{jkh} is positively homogeneous of degree zero in y^i and satisfies

(1.14) $P^i_{jkh} y^j = P^i_{kh}.$

where P^i_{jk} is called the $v(hv)-$torsion tensor and its associative tensor P_{rkh} is given by

(1.15) $a)\ g_{ir} P^i_{kh} = P_{rkh}$ and b) $P_{rkh}\, g^{ir} = P^i_{kh}$

The associate curvature tensor P_{rjkh} is given by

(1.16) $P_{rjkh} = P^i_{jkh} g_{ir}.$

P – Ricci tensor P_{jk}, the curvature vector P_k and The curvature scalar P (of Cartan's second curvature tensor) are given by

(1.17) $P_{jk} = P^i_{jki}$

(1.18) $P_k = P^i_{ki}.$

and

(1.19) $P_k\, y^k = P.$

The $hv-$ curvature tensor P^i_{jkh} satisfies the following:

(1.20) $P^i_{jkh} - P^i_{kjh} = -S^i_{jkh|r}\, y^r.$

and

(1.21) $P^i_{jkh} - P^i_{kjh} = C^i_{kh|j} + C^i_{sj} P^s_{kh} - j/k$

II. AGENERALIZED $\mathcal{B}P-$ RECURRENT FINSLER SPACE

Let us consider a Finsler space F_n for which Cartan's second curvature tensor P^i_{jkh} satisfies the generalized recurrence property with respect to Berwald's connection parameter G^i_{kh}, i.e. characterized by the following condition

(2.1) $\mathcal{B}_l P^i_{jkh} = \lambda_l P^i_{jkh} + \mu_l (\delta^i_j g_{kh} - \delta^i_k g_{jh}), \quad P^i_{jkh} \neq 0,$

where $\mathcal{B}_l$ is Berwald's covariant differential operator with respect to x^l, λ_l and μ_l are called recurrence vectors.

Definition 2.1. *A Finsler space F_n for which Cartan's second curvature tensor P^i_{jkh} satisfies the condition (2.1), where λ_l and μ_l are non-zero covariant vectors field.* Such space satisfying the condition (2.1) will be *called a generalized $\mathcal{B}P-$ recurrent spaceanddenoted it briefly by $G(\mathcal{B}P)-RF_n$.*

Let us consider a $G(\mathcal{B}P)-RF_n$ which is characterized by the condition (2.1).

Transvecting the condition (2.1) by g_{im}, using (1.16), (1.8) and (1.3b), we get

(2.2) $\mathcal{B}_l P_{mjkh} = \lambda_l P_{mjkh} + \mu_l (g_{jm} g_{kh} - g_{km} g_{jh}) + 2P^i_{jkh} y^t \mathcal{B}_t C_{iml}.$

Thus, we conclude

Theorem 2.1. *In $G(\mathcal{B}P)-RF_n$, Berwald's covariant derivative of first order for the associate curvature tensor P_{ijkh} is given by (2.2).*

Transvecting the condition (2.1) by y^j, using (1.14), (1.7), (1.3a) and (1.1), we get

(2.3) $\mathcal{B}_l P^i_{kh} = \lambda_l P^i_{kh} + \mu_l (y^i g_{kh} - \delta^i_k y_h)$.

Transvecting the condition (2.3) by g_{ij} , using (1.15a), (1.8), (1.1) and (1.3b), we get

(2.4) $\mathcal{B}_l P_{jkh} = \lambda_l P_{jkh} + \mu_l (g_{kh} y_j - g_{kj} y_h) + 2P^i_{kh} y^t \mathcal{B}_t C_{ijl}$.

Thus, we conclude

Theorem 2.2.*In $G(\mathcal{B}P) - RF_n$, Berwald's covariant derivative of first order for the (v)hv – torsion tensor P^i_{kh} and the associative torsion tensor P_{jkh} is given by (2.3) and (2.4) respectively .*

Contracting the indices i and h in the condition (2.1), using (1.17) and (1.3b), we get

(2.5) $\mathcal{B}_l P_{jk} = \lambda_l P_{jk}$.

Contracting the indices i and h in the condition (2.3), using (1.18), (1.1) and (1.3b), we get

(2.6) $\mathcal{B}_l P_k = \lambda_l P_k$.

Transvecting the condition (2.6) by y^K, using (1.19) and (1.7), we get

(2.7) $\mathcal{B}_l P = \lambda_l P$

Thus, we conclude

Theorem 2.3.*In $G(\mathcal{B}P) - RF_n$, P – Ricci tensor P_{jk}, the curvature vector P_k and the curvature scalar P behave as recurrent .*

Taking the covariant derivative for (1.20) with respect to x^l in the sense of Berwald, we get

(2.8) $\mathcal{B}_l P^i_{jkh} - \mathcal{B}_l P^i_{kjh} = \mathcal{B}_l (-S^i_{jkh|r}\, y^r)$.

Using the condition (2.1) in (2.8), we get

(2.9) $\mathcal{B}_l (-S^i_{jkh|r}\, y^r) = \lambda_l (P^i_{jkh} - P^i_{kjh}) + 2\mu_l (\delta^i_k g_{jh} - \delta^i_j g_{kh})$

Using(1.20) in (2.9), we get

(2.10) $\mathcal{B}_l (S^i_{jkh|r}\, y^r) = \lambda_l (S^i_{jkh|r}\, y^r) + \alpha_l (\delta^i_j g_{kh} - \delta^i_k g_{jh})$

Where $\alpha = 2\mu_l$

Thus, we conclude

Theorem 2.2.4.*In $G(\mathcal{B}P) - RF_n$, the tensor $(S^i_{jkh|r}\, y^r)$ is generalized recurrent.*

Taking the covariant derivative for (1.21) with respect to x^l in the sense of Berwald, we get

(2.11) $\mathcal{B}_l P^i_{jkh} - \mathcal{B}_l P^i_{kjh} = \mathcal{B}_l (C^i_{kh|j} + C^i_{sj} P^s_{kh} - j/k)$.

Using the condition (2.1) in (2.8), we get

(2.12) $\mathcal{B}_l (C^i_{kh|j} + C^i_{sj} P^s_{kh} - j/k) = \lambda_l (P^i_{jkh} - P^i_{kjh}) - 2\mu_l (\delta^i_j g_{kh} - \delta^i_k g_{jh})$.

Using(1.21) in (2.12), we get

(2.13$\mathcal{B}_l (C^i_{kh|j} + C^i_{sj} P^s_{kh} - j/k) = \lambda_l (C^i_{kh|j} + C^i_{sj} P^s_{kh} - j/k) + \alpha_l (\delta^i_j g_{kh} - \delta^i_k g_{jh})$

Where $\alpha = -2\mu_l$

Thus, we conclude

Theorem 2.5.*In $G(\mathcal{B}P) - RF_n$, the tensor $(C^i_{kh|j} + C^i_{sj} P^s_{kh} - j/k)$ is generalized recurrent.*

III. CERTAIN IDENTITIES

We shall obtain some identities which are satisfying in $G(\mathcal{B}P) - RF_n$

Using (1.9) inBianchi identity (1.11), we get

(3.1) $R^r_{ijk|h} + R^r_{ihj|k} + R^r_{ikh|j} = -(H^s_{kh} P^r_{ijs} + H^s_{jk} P^r_{ihs} + H^s_{hj} P^r_{iks})$,

Taking the covariant derivative for (3.1) with respect to x^l in the sense of Berwald, we get

(3.2) $\mathcal{B}_l (R^r_{ijk|h} + R^r_{ihj|k} + R^r_{ikh|j}) = -\{(\mathcal{B}_l H^s_{kh}) P^r_{ijs} + H^s_{kh} (\mathcal{B}_l P^r_{ijs}) + (\mathcal{B}_l H^s_{jk}) P^r_{ihs} + H^s_{jk} (\mathcal{B}_l P^r_{ihs}) + (\mathcal{B}_l H^s_{hj}) P^r_{iks} + H^s_{hj} (\mathcal{B}_l P^r_{iks}\}$

Using the condition (2.1) in (3.2), we get

(3.3) $\mathcal{B}_l (R^r_{ijk|h} + R^r_{ihj|k} + R^r_{ikh|j}) = -\lambda_l (H^s_{kh} P^r_{ijs} + H^s_{jk} P^r_{ihs} + H^s_{hj} P^r_{iks}) - \mu_l [H^s_{kh} (\delta^r_i g_{js} - \delta^r_j g_{is}) + H^s_{jk} (\delta^r_i g_{hs} - \delta^r_h g_{is}) + H^s_{hj} (\delta^r_i g_{ks} - \delta^r_k g_{is})] - [(\mathcal{B}_l H^s_{kh}) P^r_{ijs}) + (\mathcal{B}_l H^s_{jk}) P^r_{ihs} + (\mathcal{B}_l H^s_{hj}) P^r_{iks}]$

Using (3.1) in (3.3), we get

(3.4)$\mathcal{B}_l (R^r_{ijk|h} + R^r_{ihj|k} + R^r_{ikh|j}) = \lambda_l (R^r_{ijk|h} + R^r_{ihj|k} + R^r_{ikh|j}) - \mu_l [H^s_{kh} (\delta^r_i g_{js} - \delta^r_j g_{is}) + H^s_{jk} (\delta^r_i g_{hs} - \delta^r_h g_{is}) + H^s_{hj} (\delta^r_i g_{ks} - \delta^r_k g_{is})] - [(\mathcal{B}_l H^s_{kh}) P^r_{ijs} + (\mathcal{B}_l H^s_{jk}) P^r_{ihs} + (\mathcal{B}_l H^s_{hj}) P^r_{iks}]$

This shows that

(3.5) $\mathcal{B}_l (R^r_{ijk|h} + R^r_{ihj|k} + R^r_{ikh|j}) = \lambda_l (R^r_{ijk|h} + R^r_{ihj|k} + R^r_{ikh|j})$

If and only if

(3.6)
$\mu_l[H^s_{kh}(\delta^r_i g_{js} - \delta^r_j g_{is}) + H^s_{jk}(\delta^r_i g_{hs} - \delta^r_h g_{is}) + H^s_{hj}(\delta^r_i g_{ks} - \delta^r_k g_{is})] + [(\mathcal{B}_l H^s_{kh})P^r_{ijs} + (\mathcal{B}_l H^s_{jk})P^r_{ihs} + (\mathcal{B}_l H^s_{hj})P^r_{iks}] = 0$

Transvecting(3.1) byg_{rm}, using (1.10), (1.5a), and (1.16), we get
(3.7) $R_{mijk\,|h} + R_{mihj\,|k} + R_{mikh\,|j} = -(H^s_{kh}P_{mijs} + H^s_{jk}P_{mihs} + H^s_{hj}P_{miks})$,
Taking the covariant derivative for (3.7) with respect to x^l in the sense of Berwald, we get
(3.8) $\mathcal{B}_l(R_{mijk\,|h} + R_{mihj\,|k} + R_{mikh\,|j}) = -\{(\mathcal{B}_l H^s_{kh})P_{mijs} + H^s_{kh}(\mathcal{B}_l P_{mijs}) + (\mathcal{B}_l H^s_{jk})P_{mihs} + H^s_{jk}(\mathcal{B}_l P_{mihs}) + (\mathcal{B}_l H^s_{hj})P_{miks} + H^s_{hj}(\mathcal{B}_l P_{miks})\}$
Using the condition (2.2) in (3.8), we get
(3.9) $\mathcal{B}_l(R_{mijk\,|h} + R_{mihj\,|k} + R_{mikh\,|j}) = -\lambda_l(H^s_{kh}P_{mijs} + H^s_{jk}P_{mihs} + H^s_{hj}P_{miks}) - \mu_l[H^s_{kh}(g_{mi}g_{js} - g_{mj}g_{is}) + H^s_{jk}(g_{mi}g_{hs} - g_{mh}g_{is}) + H^s_{hj}(g_{mi}g_{ks} - g_{mk}g_{is})] - [(\mathcal{B}_l H^s_{kh})P_{mijs}) + (\mathcal{B}_l H^s_{jk})P_{mihs} + (\mathcal{B}_l H^s_{hj})P_{miks}] + 2y^t\mathcal{B}_t C_{iml}[P^r_{ijs} + P^r_{ihs} + P^r_{iks}]$
Using (3.7) in (3.9), we get
(3.10)$\mathcal{B}_l(R_{mijk\,|h} + R_{mihj\,|k} + R_{mikh\,|j}) = \lambda_l(R_{mijk\,|h} + R_{mihj\,|k} + R_{mikh\,|j}) - \mu_l[H^s_{kh}(g_{mi}g_{js} - g_{mj}g_{is}) + H^s_{jk}(g_{mi}g_{hs} - g_{mh}g_{is}) + H^s_{hj}(g_{mi}g_{ks} - g_{mk}g_{is})] - [(\mathcal{B}_l H^s_{kh})P_{mijs}) + (\mathcal{B}_l H^s_{jk})P_{mihs} + (\mathcal{B}_l H^s_{hj})P_{miks}] + 2y^t\mathcal{B}_t C_{iml}[P^r_{ijs} + P^r_{ihs} + P^r_{iks}]$

This shows that
(3.11) $\mathcal{B}_l(R_{mijk\,|h} + R_{mihj\,|k} + R_{mikh\,|j}) = \lambda_l(R_{mijk\,|h} + R_{mihj\,|k} + R_{mikh\,|j})$
If and only if
(3.12) $\mu_l[H^s_{kh}(g_{mi}g_{js} - g_{mj}g_{is}) + H^s_{jk}(g_{mi}g_{hs} - g_{mh}g_{is}) + H^s_{hj}(g_{mi}g_{ks} - g_{mk}g_{is})] + [(\mathcal{B}_l H^s_{kh})P_{mijs}) + (\mathcal{B}_l H^s_{jk})P_{mihs} + (\mathcal{B}_l H^s_{hj})P_{miks}] + 2y^t\mathcal{B}_t C_{iml}[P^r_{ijs} + P^r_{ihs} + P^r_{iks}] = 0$

Transvecting(3.1) byy^i, using (1.9), (1.6a) and (1.14), we get
(3.13) $H^r_{jk\,|h} + H^r_{hj\,|k} + H^r_{kh|j} = -(H^s_{kh}P^r_{js} + H^s_{jk}P^r_{hs} + H^s_{hj}P^r_{ks})$
Taking the covariant derivative for (3.13) with respect to x^l in the sense of Berwald, we get
(3.14) $\mathcal{B}_l(H^r_{jk\,|h} + H^r_{hj\,|k} + H^r_{kh|j} = -\{(\mathcal{B}_l H^s_{kh})P^r_{js} + H^s_{kh}(\mathcal{B}_l P^r_{js}) + (\mathcal{B}_l H^s_{jk})P^r_{hs} + H^s_{jk}(\mathcal{B}_l P^r_{hs}) + (\mathcal{B}_l H^s_{hj})P^r_{ks} + H^s_{hj}(\mathcal{B}_l P^r_{ks}\}$
Using the condition (2.3) in (3.14), we get
(3.15) $\mathcal{B}_l(H^r_{jk\,|h} + H^r_{hj\,|k} + H^r_{kh|j}) = -\lambda_l(H^s_{kh}P^r_{js} + H^s_{jk}P^r_{hs} + H^s_{hj}P^r_{ks}) - \mu_l[H^s_{kh}(y^r g_{js} - \delta^r_j y_s) + H^s_{jk}(y^r g_{hs} - \delta^r_h y_s) + H^s_{hj}(y^r g_{ks} - \delta^r_k y_s)] - [(\mathcal{B}_l H^s_{kh})P^r_{js}) + (\mathcal{B}_l H^s_{jk})P^r_{hs} + (\mathcal{B}_l H^s_{hj})P^r_{ks}]$
Using (3.13) in (3.15), we get
(3.16) $\mathcal{B}_l(H^r_{jk\,|h} + H^r_{hj\,|k} + H^r_{kh|j}) = \lambda_l(H^r_{jk\,|h} + H^r_{hj\,|k} + H^r_{kh|j}) - \mu_l[H^s_{kh}(y^r g_{js} - \delta^r_j y_s) + H^s_{jk}(y^r g_{hs} - \delta^r_h y_s) + H^s_{hj}(y^r g_{ks} - \delta^r_k y_s)] - [(\mathcal{B}_l H^s_{kh})P^r_{js}) + (\mathcal{B}_l H^s_{jk})P^r_{hs} + (\mathcal{B}_l H^s_{hj})P^r_{ks}]$
This shows that
(3.17) $\mathcal{B}_l(H^r_{jk\,|h} + H^r_{hj\,|k} + H^r_{kh|j}) = \lambda_l(H^r_{jk\,|h} + H^r_{hj\,|k} + H^r_{kh|j})$
If and only if
(3.18) $\mu_l[H^s_{kh}(y^r g_{js} - \delta^r_j y_s) + H^s_{jk}(y^r g_{hs} - \delta^r_h y_s) + H^s_{hj}(y^r g_{ks} - \delta^r_k y_s)] + [(\mathcal{B}_l H^s_{kh})P^r_{js}) + (\mathcal{B}_l H^s_{jk})P^r_{hs} + (\mathcal{B}_l H^s_{hj})P^r_{ks}] = 0$
The equation (3.5), (3.11) and (3.17) show that the tensors $(R^r_{ijk\,|h} + R^r_{ihj\,|k} + R^r_{ikh|j})$, $(R_{mijk\,|h} + R_{mihj\,|k} + R_{mikh\,|j})$ and $(H^r_{jk\,|h} + H^r_{hj\,|k} + H^r_{kh|j})$ behave as recurrent if and only if (3.6), (3.12) and (3.18) respectively holds.
Thus, we conclude

Theorem 3.1.*In $G(\mathcal{B}P) - RF_n$, tensors $(R^r_{ijk\,|h} + R^r_{ihj\,|k} + R^r_{ikh|j})$, $(R_{mijk\,|h} + R_{mihj\,|k} + R_{mikh\,|j})$ and $(H^r_{jk\,|h} + H^r_{hj\,|k} + H^r_{kh|j})$ behave as recurrent if and only if (3.6), (3.7) and (3.18) respectively holds.*

IV. THE PROJECTION ON INDICATRIX IN G($\mathcal{B}R$) $- RF_n$

Definition 4.1.*Let the current coordinates in the tangent space at the pointx_0 bex^i, then the indicatrixI_{n-1} is a hypersurface defined by*[8]
(4.1) $F(x_0, x^i) = 1$
or in the parametric form it is defined by
(4.2)$x^i = x^i(u^a)$, $a = 1,2,\dots,n-1$.

Definition 4.2. *The projection of any tensor* T^i_j *on the indicatrix* I_{n-1} *is given by* [7]

(4.3) $p.T^i_j := T^a_b \, h^i_a \, h^b_j$,

where

(4.4) $h^i_c := \delta^i_c - l^i l_c$.

The projection of the vector y^i, *the unit vector* l^i *and the metric tensor* g_{ij} *on the indicatrix are given by* [8]

(4.5) a) $p.y^i = 0$,

b) $p.l^i = 0$

and

c) $p.\, g_{ij} = h_{ij}$,

where

(4.6) $h_{ij} := g_{ij} - l_i l_j$.

Our aim is to discuss the projection of the generalized recurrence property on indicatrix with respect to Berwald's connection for the tensors whose behave as recurrent in $G(\mathcal{B}P) - RF_n$. We obtained some theorems in projection on indicatrix for Berwald's curvature tensor, the P–Ricci tensor P_{jk} and the curvature vector P_k .

Let us consider $(\mathcal{B}P) - RF_n$, for which the $P-$ Ricci tensor P_{jk} behaves as recurrent in the sense of Berwald, i.e. satisfied (2.5).

In view of (4.3), the projection of the $P-$ Ricci tensor P_{jk} on the indicatrix is given by

(4.7) $p.P_{jk} = P_{ab} \, h^a_j h^b_k$.

Taking the covariant derivative for the equation (4.7) with respect to x^l, in the sense of Berwald, we get

(4.8) $\mathcal{B}_l(p.P_{jk}) = \mathcal{B}_l(P_{ab} \, h^a_j h^b_k)$.

Using (2.5) and the fact that h^a_b is covariant constant in (4.8), we get

(4.9) $\mathcal{B}_l(p.P_{jk}) = \lambda_l P_{ab} \, h^a_j h^b_k$.

Using (4.7) in (4.9), we get

(4.10) $\mathcal{B}_l(p.P_{jk}) = \lambda_l(p.P_{jk})$.

Thus, we conclude

Theorem 4.1. *In* $(\mathcal{B}P) - RF_n$ *, the projection of the P –Ricci tensor* P_{jk} *on indicatrix is recurrent .*

Let us consider $(\mathcal{B}R) - RF_n$, for which the curvature vector P_k behaves as recurrent in the sense of Berwald, i.e. satisfied (2.6).

In view of (4.3), the projection of the curvature vector P_k on the indicatrix is given by

(4.11) $p.P_k = P_a h^a_k$.

Taking the covariant derivative for the equation (4.11) with respect to x^l, in the sense of Berwald, we get

(4.12) $\mathcal{B}_l(p.P_k) = \mathcal{B}_l(P_a h^a_k)$.

Using (2.6) and the fact that h^a_b is covariant constant in (4.12), we get

(4.13) $\mathcal{B}_l(p.P_k) = \lambda_l P_a h^a_k$.

Using (4.11) in (4.13), we get

(4.14) $\mathcal{B}_l(p.P_k) = \lambda_l(p.P_k)$.

Thus, we conclude

Theorem 4.2. *In* $(\mathcal{B}P) - RF_n$ *, the projection of the curvature vector* P_k *on indicatrix is recurrent .*

Let us consider a Finsler space F_n for which the projection of the $P-$ Ricci tensor P_{jk} on indicatrixbehaves as recurrent with respect to Berwald's connection, i.e. satisfied the equation (4.10).

Using (4.3) in (4.10), we get

(4.15) $\mathcal{B}_m(P_{ab} \, h^a_j h^b_k) = \lambda_m(P_{ab} \, h^a_j h^b_k)$.

Using (4.4) in (4.15), we get

(4.16) $\mathcal{B}_m\{P_{ab}\left(\delta^a_j \delta^b_k - \delta^a_j l^b l_k - l^a l_j \delta^b_k + l^a l_j l^b l_k\right)\} = \lambda_m \{P_{ab}\left(\delta^a_j \delta^b_k - \delta^a_j l^b l_k - l^a l_j \delta^b_k + l^a l_j l^b l_k\right)\}$.

Using (1.4) in (4.16), we get

(4.17) $\mathcal{B}_m\left(P_{jk} - P_{jb}\frac{1}{F}y^b l_k - P_{ak}\frac{1}{F}y^a l_j + P_{ab}\frac{1}{F^2}y^a y^b l_j l_k\right)$

$= \lambda_m\left(P_{jk} - P_{jb}\frac{1}{F}y^b l_k - P_{ak}\frac{1}{F}y^a l_j + P_{ab}\frac{1}{F^2}y^a y^b l_j l_k\right)$.

Now, if $P_{jb}\, y^b = 0 = P_{ak}\, y^a$, then the equation (4.17) becomes

(4.18) $\mathcal{B}_m P_{jk} = \lambda_m P_{jk}$.

Thus, we conclude

Theorem 4.3. *If theprojection of the* $P-$ *Ricci tensor* P_{jk} *of* $G(\mathcal{B}R) - RF_n$ *on indicatrix is recurrent, then the* $P-$ *Ricci tensor* P_{jk} *itself recurrent, provided* $P_{jb}\, y^b = 0 = P_{ak}\, y^a$.

Let us consider a Finsler space F_n for which the projection of the curvature vector P_k on indicatrixbehaves as recurrent with respect to Berwald's connection, i.e. satisfied the equation (4.14).

Using (4.3) in (4.14), we get

(4.19) $\mathcal{B}_m (P_a h_k^a) = \lambda_m (P_a h_k^a)$.

Using (4.4) in (4.19), we get

(4.20) $\mathcal{B}_m \{P_a (\delta_k^a - \mathfrak{l}^a \mathfrak{l}_k)\} = \lambda_m \{P_a (\delta_k^a - \mathfrak{l}^a \mathfrak{l}_k)\}$.

Using (1.4) in (4.22), we get

(4.21) $\mathcal{B}_m \left(P_k - P_a \frac{1}{F} y^a \mathfrak{l}_k\right) = \lambda_m \left(P_k - P_a \frac{1}{F} y^a \mathfrak{l}_k\right)$.

Now, if $P_a\, y^a = 0$, then the equation (4.21) becomes

(4.22) $\mathcal{B}_m P_k = \lambda_m P_k$.

Thus, we conclude

Theorem 4.4. *If theprojection of the curvature vector P_k of $G(\mathcal{B}R) - RF_n$ on indicatrix is recurrent, then the curvature vector P_k itself recurrent, provided $P_a\, y^a = 0$.*

V. CONCLUSION

(1) The space which defined by the condition (2.1) we called a generalized $\mathcal{B}P -$ recurrent Finsler space.

(2) In generalized $\mathcal{B}P -$ recurrent Finsler space ,P – Ricci tensor P_{jk} , the curvature vector P_k and the curvature scalar P behave as recurrent.

(3) In generalized $\mathcal{B}P -$ recurrent Finsler space, Berwald's covariant derivative of first order for the associate curvature tensor P_{ijkh} , the *(v)hv–* torsion tensor P_{kh}^i and the associative torsion tensor P_{jkh} are non-vanishing.

(4) In generalized $\mathcal{B}P -$ recurrent Finsler space, the tensor $\left(S_{jkh|r}^i\, y^r\right)$ and the tensor $(C_{kh|j}^i + C_{sj}^i P_{kh}^s - j/k)$ are generalized recurrent.

(5) In generalized $\mathcal{B}P -$ recurrent Finsler space, the tensor $(R_{ijk|h}^r + R_{ihj|k}^r + R_{ikh|j}^r)$, $(R_{mijk|h} + R_{mihj|k} + R_{mikh|j})$ and $\left(H_{jk|h}^r + H_{hj|k}^r + H_{kh|j}^r\right)$ behave as recurrent if and only if (3.6), (3.7) and (3.18) respectively holds.

(6) We obtained some certain identities in generalized $\mathcal{B}P -$ recurrent Finsler space.

(7) In generalized $\mathcal{B}P -$ recurrent Finsler space the projection of the P – Ricci tensor P_{jk} and curvature vector P_k on indicatrix is recurrent .

(8) If the projection of $P -$ Ricci tensor P_{jk} and the projection of curvature vector P_k of $G(\mathcal{B}R) - RF_n$ on indicatrix is recurrent, then the $P -$ Ricci tensor P_{jk} and curvature vector P_k itself recurrent, provided $P_{jb}\, y^b = 0 = P_{ak}\, y^a$ and $P_a\, y^a = 0$ respectively.

REFERENCES

[1] A.H.Mohammed,On study of P^h- recurrent curvature tensors in different types of Finsler space, M.Sc. Thesis, University of Aden, (Aden) (Yemen), (2016).

[2] A.H.Awed,On study of generalized P^h- recurrent Finsler space, M.Sc. Thesis, University of Aden, (Aden) (Yemen), (2017).

[3] A.M.Hanballa,On covariant differentiation of curvature tensors in Finsler spaces, Ph.D. Thesis, Faculty of Education - Aden, University of Aden, (Aden) (Yemen), (2016).

[4] A. Zafar and A.Musavvir,On some properties of W – curvature tensor, Palestine Journal of Mathematices, Vol.3(1), (2014), 61 – 69.

[5] F.Y.Qasem,On transformations in Finsler spaces,D. Phil. Thesis, University of Allahabad, (Allahabad) (India) (2000).

[6] F.Y.Qasem and A.A.Abdallah, On certain generalized $\mathcal{B}R$ –recurrent Finsler space, International Journal of Applied Science and Mathematics, Volume 3, Issue 3, (2016), 111-114 .

[7] F.Y.Qasem and S.M.Baleedi,On a generalized $\mathcal{B}K$ –recurrent Finsler space, International Journal of Science basic and applied reserch, Volume 28, No.3, (2016), 195-203.

[8] H. Rund,TheDifferential Geometry of FinslerSpaces, Springer-Verlag, Berlin Göttingen, 2nd Edit. Nauka, Moscow-Russian, 1959.

[9] P.N.Pandey, Some problems in Finsler spaces, D.Sc. Thesis, University of Allahabad, (Allahabad) (India) , (1993) .

[10] P.N. Pandey, S. Saxena and A. Goswani, On a generalized H – recurrent space, Journal of International Academy of Physical Science, Vol.15, (2011), 201 – 211.

[11] R. Verma, Some transformations in Finsler space, Ph.D.Thesis, University of Allahabad, (Allahabad), (India), (1991).

[12] S. Dikshit, Certain types of recurrences in Finsler spaces, D. phil. Thesis, University of Allahabad, (Allahabad) (India) , (1992).