Degree of Approximation of Function in the Holder Metric by (N,Pn) (E,q) Means

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Abstract
In this paper, a theorem on degree of approximation of function in the Holder metric by (N,Pn) (E,q) means has been established.

Keywords - *Degree of approximation, Holder metric, (N,Pn) mean, (E,q) mean.*

I. INTRODUCTION

The degree of approximation of a function f belonging to various classes using different Summability method has been determined by many Mathematician ,Chandra[1] find the degree of approximation of function by Norlund transform .Later on Mahapatra and Chandra[2]obtain the degree of approximation in Holder metric using matrix transform .I n sequal singh et.al. [8] obtain the error bound of periodic function in Holder metric again Mishra et.al. gave the generalization of result of Singh et.al. In this paper we find the degree of approximation of function in Holder metric by (N,Pn) (E,q) means.

II. DEFINITION

Let f be a periodic function of period 2π integrable in the sense of Lebesgue over[π, - π]. Let the Fourier series of f given by

$$f(t) \approx \frac{a_o}{2} + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx) \qquad \text{.......(2.1)}$$

Let $c_{2\pi}$ denote the Banach Space of all 2π - periodic continuous function defined on $[\pi, -\pi]$ under sub-norm. For 0 ≤ α ≤1 and some positive constant k the function space H_α is given by the following

$$H_\alpha = \{f \in c_{2\pi} : |f(x) - f(y)| \le k|x-y|^\alpha \} \qquad . \qquad \text{.......(2.2)}$$

The space H_α is a Banach space with the norm $\|.\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_c + \sup_{x,y} [\Delta^\alpha f(x,y)] \qquad \text{..........(2.3)}$$

Where $\|f\|_c = \sup_{-\pi \le x \le \pi} |f(x)|$ and $\Delta^\alpha f(x,y) = \frac{|f(x)-f(y)|}{|x-y|^\alpha}$ $x \ne y$. We shall use the connection that $\Delta^0 f(x,y) = 0$.

The metric induced by norm in (2.3) on H_α is called the Holder metric. We write *through the paper*

$$\emptyset_x(t) = f(x+t) - 2f(x) + f(x-t) \qquad \text{.......(2.4)}$$

$$K_n(t) = \frac{1}{2\pi P_n} \sum_{k=o}^{n} \frac{p_n}{(1+q)^k} \left\{ \sum_{v=0}^{k} \binom{k}{v} q^{k-v} \frac{\sin(v+\frac{1}{2})t}{\sin(\frac{t}{2})} \right\} \qquad \text{..........(2.5)}$$

III. KNOWN RESULTS

In 1982 Mahapatra and Chandra [1] considered the $E_n^q(f,x)$ for the holder continuous function f to obtain error bounds in Holder norm. They proved the following

Theorem – Let $0 \le \beta < \alpha \le 1$ and let $f \in H_\alpha$ then for $n > 1$

$$\|f - E_n^q(f)\|_\beta = o\left[(n)^{\frac{-(\alpha-\beta)}{2}} (\log n)^{\frac{\beta}{\alpha}}\right]. \quad .....(3.1)$$

Above theorem improved by Chandra [5] in 1988 and proved

Theorem – Let $0 \le \beta < \alpha \le 1$ and let $f \in H_\alpha$ then for $n > 1$

$$\|f - E_n^q(f)\|_\beta = o\left[(n)^{\beta-\alpha} (\log n)^{\frac{\beta}{\alpha}}\right]. \quad .....(3.2)$$

Singh and Mahajan [8] established the following theorem to error bound of signal passing through (C,1)(E,1) transform.

Theorem 1 – Let $w(t)$ defined (2.4) be such that

$$\int_t^\pi \frac{w(u)}{u^2} du = o\{H(t)\} \qquad H(t) \ge 0 \quad ....(3.3)$$

$$\int_0^t H(u) du = o\{tH(t)\} \qquad as\ t \to 0^+ \quad ....(3.4)$$

Then for $0 \le \beta < \alpha \le 1$ and $f \in H_w$ we have

$$\left\|t_n^{(CE)^1}(S; f) - f(x)\right\|_{w^*} = o\left\{\left((n+1)^{-1} H(\tfrac{\pi}{n+1})\right)^{1-\frac{\beta}{\alpha}}\right\} \quad ....(3.5)$$

Theorem 2 – Consider w(t) defined (2.4) and for $0 \le \beta \le \alpha \le 1$ and $f \in H_w$ we have

$$\left\|t_n^{(CE)^1}(f) - f(x)\right\|_{w^*} = o\left\{\left(w(\tfrac{\pi}{n+1})\right)^{1-\frac{\beta}{\alpha}} + ((n+1)^{-1} \textstyle\sum_{k=1}^{n+1} w\left(\frac{1}{k+1}\right))^{1-\frac{\beta}{\alpha}}\right\}$$

.........(3.6)

In sequal Mishra and Khatri [10] gave the generalized result of above theorem . They proved the following.

Theorem 3 – Let $w(t)$ defined (2.4) be such that

$$\int_t^\pi \frac{w(u)}{u^2} du = o\{H(t)\} \qquad H(t) \ge 0$$

$$\int_0^t H(u) du = o\{tH(t)\} \qquad as\ t \to 0^+$$

Let Np be the Norlund summability matrix generated by the non –negative {Pn} such that (n+1) pn = o(Pn) $\forall n \ge 0$.

Then for $\bar{f} \in H_w$ $0 \le \beta < \alpha \le 1$ we have

$$\|t_n^{-NE}(f) - \bar{f}(x)\| w^* = o\left\{\frac{w(|x-y|)^{\frac{\beta}{\alpha}}}{\omega^*(|x-y|)} (\log(n+1))^{\frac{\beta}{\alpha}} \left((n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right)^{1-\frac{\beta}{\sigma}}\right\} \quad ....(3.7)$$

And if w(t) satisfies (3.1) then for $\bar{f} \in H_w$ $0 \le \beta < \alpha \le 1$ we have

$$\|t_n^{-NE}(f) - \bar{f}(x)\| w^* = o\left\{\frac{w(|x-y|)^{\frac{\beta}{\alpha}}}{\omega^*(|x-y|)} \left(\log(n+1)\, w(\tfrac{\pi}{n+1})\right)^{1-\frac{\beta}{\alpha}} + \left(\left(\frac{1}{n+1}\right) \textstyle\sum_{k=0}^{n} w\left(\frac{\pi}{n+1}\right)\right)^{1-\frac{\beta}{\sigma}}\right\}$$

.......(3.8)

IV. MAIN RESULT

In this paper we prove the following theorem

Theorem – For $0 \le \beta < \alpha \le 1$ *and* $f \in H_\alpha$ then let $f \in H_\alpha$ then for $n > 1$

$$\|t_n(f) - f\|_\beta = o\left\{n^{\beta-\alpha} \log n^{\left(\frac{\beta}{\alpha}\right)}\right\} \quad \ldots\ldots\ldots(4.1)$$

V. LEMMA

Lemma 5(a) - If $\emptyset_x(t)$ defined in (2.5) then for $f \in H_\alpha$ *and* $0 < \alpha \le 1$ we have

$$\left|\emptyset_x(t) - \emptyset_y(t)\right| = M(|x-y|^\alpha) \quad \ldots\ldots..(5.1)$$

$$\left|\emptyset_x(t) - \emptyset_y(t)\right| = M(|t|^\alpha) \quad \ldots\ldots(5.2)$$

Lemma 5(b) - For $0 \le t \le \frac{\pi}{n}$ we have $\sin nt = n\, sint$

$$|K_n(t)| = o(n) \quad \ldots\ldots(5.3)$$

Proof - For $0 \le t \le \frac{\pi}{n}$ and $\sin nt = n\, sint$ then

$$|K_n(t)| = \left|\frac{1}{2\pi P_n} \sum_{k=o}^{n} \frac{p_{n-k}}{(1+q)^k} \left\{\sum_{v=0}^{k} \binom{k}{v} q^{k-v} \frac{\sin(v+\frac{1}{2})t}{\sin(\frac{t}{2})}\right\}\right|$$

$$\le \frac{1}{2\pi P_n} \left|\sum_{k=o}^{n} \frac{p_{n-k}}{(1+q)^k} \left\{\sum_{v=0}^{k} \binom{k}{v} q^{k-v} \frac{(2v+1)\sin(\frac{t}{2})}{\sin(\frac{t}{2})}\right\}\right|$$

$$\le \frac{1}{2\pi P_n} \left|\sum_{k=o}^{n} \frac{p_{n-k}}{(1+q)^k} (2k+1)\left\{\sum_{v=0}^{k} \binom{k}{v} q^{k-v}\right\}\right|$$

$$\le \frac{1}{2\pi P_n} \left|\sum_{k=o}^{n} p_{n-k}(2k+1)\right|$$

$$= \frac{(2n+1)}{2\pi P_n} \left|\sum_{k=o}^{n} p_{n-k}\right|$$

$$= o(n)$$

Lemma 5(c) - For $\frac{\pi}{n} \le t \le \pi$, $sin\frac{t}{2} \ge \frac{t}{\pi}$ *and* $sin\, nt \le 1$ we have

$$|K_n(t)| = o\left(\frac{1}{t}\right) \quad \ldots\ldots\ldots\ldots(5.4)$$

Proof - For $\frac{\pi}{n} \le t \le \pi$, $sin\frac{t}{2} \ge \frac{t}{\pi}$ *and* $sin\, nt \le 1$

$$|K_n(t)| = \left|\frac{1}{2\pi P_n} \sum_{k=o}^{n} \frac{p_{n-k}}{(1+q)^k} \left\{\sum_{v=0}^{k} \binom{k}{v} q^{k-v} \frac{\sin(v+\frac{1}{2})t}{\sin(\frac{t}{2})}\right\}\right|$$

$$\le \frac{1}{2\pi P_n} \left|\sum_{k=o}^{n} \frac{p_{n-k}}{(1+q)^k} \left\{\sum_{v=0}^{k} \binom{k}{v} q^{k-v} \frac{\pi}{t}\right\}\right|$$

$$\le \frac{1}{2t P_n} \left|\sum_{k=o}^{n} p_{n-k}\right|$$

$$= o\left(\frac{1}{t}\right)$$

VI. PROOF OF THEOREM 4

Let $S_n(x)$ denotes the partial sum of fourier series given in (2.1) then we have

$$S_n(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \emptyset(t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}dt \quad \text{....(6.1)}$$

The (E,q) transform E_n^q of S_n is given by

$$E_n^q - f(x) = \frac{1}{2\pi(1+q)^n}\int_0^{\pi} \emptyset(t)\left\{\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}dt \quad \text{......(6.2)}$$

The (N,Pn) (E,q) transform of $S_n(x)$ is given by

$$t_n^{NE}(f) - f(x) = \frac{1}{2\pi P_n}\sum_{k=0}^{n}\left[\frac{p_{n-k}}{(1+q)^k}\int_0^{\pi} \emptyset(t)\left\{\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}dt\right] \quad \text{......(6.3)}$$

$$= \int_0^{\pi} \emptyset(t)\ k_n(t)$$

$$= \left[\int_0^{\frac{\pi}{n}}. + \int_{\frac{\pi}{n}}^{\pi}.\right]\emptyset(t)k_n(t) \quad \text{.....(6.4)}$$

Now $E_n(x) = |t_n^{NE}(f) - f(x)|$ and $E_n(x,y) = |E_n(x) - E_n(y)|$

$$E_n(x,y) = |E_n(x) - E_n(y)|$$

$$= \left[\int_0^{\frac{\pi}{n}}. + \int_{\frac{\pi}{n}}^{\pi}.\right]\left|\emptyset_x(t) - \emptyset_y(t)\right||k_n(t)|dt$$

$$= I_1 + I_2 \quad \text{....(6.5)}$$

Again $I_1 = \int_0^{\frac{\pi}{n}}.\left|\emptyset_x(t) - \emptyset_y(t)\right||k_n(t)|\ dt$

Using lemma (3.2) and (3.2) we get

$$= o(n)\int_0^{\frac{\pi}{n}}.t^{\alpha}\ dt$$

$$= o(n)\left\{\left(\frac{\pi}{n}\right)^{\alpha+1}\right\}$$

$$= o(n)^{-\alpha} \quad \text{.....(6.6)}$$

Now $I_2 = \int_{\frac{\pi}{n}}^{n}.\left|\emptyset_x(t) - \emptyset_y(t)\right||k_n(t)|\ dt$

$$= \int_{\frac{\pi}{n}}^{n}.t^{\alpha}\left(\frac{1}{t}\right)\ dt$$

$$= o(n)^{-\alpha} \quad \text{......(6.7)}$$

Again $I_1 = \int_0^{\frac{\pi}{n}} . |\emptyset_x(t) - \emptyset_y(t)||k_n(t)| \, dt$

$= o(|x-y|^\alpha n)$ ……(6.8)

$I_2 = \int_{\frac{\pi}{n}}^{n} . |\emptyset_x(t) - \emptyset_y(t)||k_n(t)| \, dt$

$= o|x-y|^\alpha \int_{\frac{\pi}{n}}^{n} . |k_n(t)| \, dt$

$= o|x-y|^\alpha \int_{\frac{\pi}{n}}^{n} . \left(\frac{1}{t}\right) dt$

$= o(|x-y|^\alpha \log n)$ …….(6.9)

Now $I_r = I_r^{1-\frac{\beta}{\alpha}} \quad I_r^{\frac{\beta}{\alpha}}$ r = 1,2,3, ……….

From (6.6) and (6.8) we get

$I_1 = o\left[\{(n)^{-\alpha}\}^{1-\frac{\beta}{\alpha}} \quad \{|x-y|^\alpha (n)\}^{\frac{\beta}{\alpha}}\right]$

$= o\left[(n)^{\beta-\alpha} |x-y|^\beta \quad (n)^{\frac{\beta}{\alpha}}\right]$

$= o\left[(n)^{\beta-\alpha+\frac{\beta}{\alpha}} \quad |x-y|^\beta\right]$ …….(6.10)

From (6.7) and (6.9) we get

$I_2 = o\left[\{(n)^{-\alpha}\}^{1-\frac{\beta}{\alpha}} \{|x-y|^\alpha (\log n)\}^{\frac{\beta}{\alpha}}\right]$

$= o\left[(n)^{\beta-\alpha} |x-y|^\beta (\log n)^{\frac{\beta}{\alpha}}\right]$ ……..(6.11)

Now from (6.10) and (6.11) we get

$|f(x) - f(y)| = o\left[(n)^{\beta-\alpha+\frac{\beta}{\alpha}} \quad |x-y|^\beta\right] + o\left[(n)^{\beta-\alpha} |x-y|^\beta (\log n)^{\frac{\beta}{\alpha}}\right]$

$= o\left[(n)^{\beta-\alpha} |x-y|^\beta (\log n)^{\frac{\beta}{\alpha}}\right]$

And $\Delta^\beta[f(x,y)] = \frac{|f(x)-f(y)|}{|x-y|^\beta}$ $(x \neq y)$

$= o\left[(n)^{\beta-\alpha} \quad (\log n)^{\frac{\beta}{\alpha}}\right]$ …….(6.12)

Now $\|f\|_c = o[(n)^{-\alpha}]$ ……(6.13)

Combining (6.12) and (6.13) we get

$\|t_n(f) - f\|_\beta = o\left[(n)^{\beta-\alpha} \quad (\log n)^{\frac{\beta}{\alpha}}\right].$

This complete the proof of theorem.

ACKNOWLEDGEMENT

The author would like to express their deep gratitude to the anonymous learned refree(s) for their Valuable suggestion and constructive comments which resulted in the subsequent improvement of This paper.

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