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Duplication of some graph elements and absolute mean graceful labeling

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Abstract :

In the present paper, we derived absolute mean graceful labeling for some graphs obtained by duplication of graph elements in complete bipartite graph $K_{m,n}$, cycle C_n, path P_n and swastik graph Sw_n.

Key Words : Duplication of a vertex by vertex, duplication of a vertex by edge, duplication of edge by a vertex, α - absolute mean graceful graph, swastik graph.

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1. Introduction :

All the graphs, going to be discussed in the present paper, are finite, simple and undirected. Let G be a (p, q) graph. For a graph $G = (V, E)$ a function having domain V or E or $V \cup E$ is known as graph labeling for G. Throughout this paper, complete bipartite graph will be denoted as $K_{m,n}$ with m part $M = \{ u_1, u_2, ..., u_m \}$ and n part $N = \{ v_1, v_2, ..., v_n \}$. i.e. $V(K_{m,n}) = M \cup N$. Concept of graph labeling was invented by Rosa [1]. Kaneria and Chudasama [2] gave more freedom to graceful labeling and introduced absolute mean graceful labeling. Kaneria and Chudasama [3] proved that graceful labeling is preserved even after duplication by vertex or edge in $K_{m,n}$. In this present work, we obtained absolute mean graceful labeling for various graphs obtained by duplication of same graph elements. We also proved that Swastik graph is absolute mean graceful which was earlier proved graceful graph by Kaneria and Makadia [4]. For conceptual study and notations, we referred Gallian [5] and Harary [6].

Duplication of a vertex v of a graph G is the graph G' by adding a new vertex v' (duplicant of v) such that $N_{G'}(v') = N_G(v) = N_{G'}(v)$. i.e. v' is adjacent with all vertices of G which are adjacent to v in G.

Duplication of a vertex v by a new edge $e = v'v''$ in a graph G produces a new graph $G' = (\ V(G) \cup \{v', v''\}\ ,\ E(G) \cup \{v'v'', vv', vv''\}\ )$. Duplication of an edge $e = uv$ by a new vertex w in a graph G produces a new graph $G' = (\ V(G) \cup \{w\}, E(G) \cup \{uw, vw\})$. i.e. $N_{G'}(w) = \{u, v\}$.

Swastik graph is an union of four copies on C_{4n}. If $v_{i,j}$ ($\forall\ i = 1$, 2, 3, 4; $\forall\ j = 1$, 2,..., $4n$) be vertices of i^{th} copy of $C_{4n}^{(i)}$, the we shall combine $v_{1,4t}$ and $v_{2,1}$; $v_{2,4t}$ and $v_{3,1}$; $v_{3,4t}$ and $v_{4,1}$; $v_{4,4t}$ and $v_{1,1}$ by four vertices. So graph seems like a plus sign. If we bend branches of graph toward clockwise at the middle, then the graph looks a swastik which is denoted as Sw_n of n size, where $n \in N - \{1\}$. Clearly, $\mid V(Sw_n) \mid = 16(n) - 4$ and $\mid E(Sw_n) \mid = 16(n)$.

Definition : α - absolute mean graceful graph

A function f is called an absolute mean graceful labeling of a graph $G = (V, E)$, if $f : V(G) \to \{0, \pm1, \pm2, ..., \pm q\}$ is injective and the induced function $f^* : E(G) \to \{1, 2, ..., q\}$ defined as $f^*(e) = \left\lceil \frac{\mid f(u) - f(v) \mid}{2} \right\rceil$ is bijective for every edge $e = (u, v) \in E(G)$. A labeling f is said to be α - labeling, if there exists an integer k such that for each edge uv either $f(u) \leq k <\mid f(v) \mid$ or $f(v) \leq k <\mid f(u) \mid$, $\forall u, v \in V(G)$. The graph which holds absolute mean graceful labeling and α - labeling is called α - absolute mean graceful graph.

2. Main Results :

Theorem 2.1 : Duplication of any vertex in $K_{m,n}$ is α - absolute mean graceful graph.

***Proof*:** Let G be a graph obtained by duplication of one vertex of $K_{m,n}$. It is obvious that G is either $K_{m+1,n}$ or $K_{m,n+1}$. Since $K_{m,n}$, $\forall\ m, n \in N$ is absolute mean graceful graph proved by Kaneria and Chudasama [2], $K_{m+1,n}$ and $K_{m,n+1}$ both are absolute mean graceful graphs. Hence, G is absolute mean graceful graph. It also satisfies α-labeling. So, G is α - absolute mean graceful graph. □

Theorem 2.2 : Duplication of all the vertices of m-part or n-part in $K_{m,n}$ is

α - absolute mean graceful graph.

***Proof*:** Let G be a graph obtained by duplication of all the vertices of $M = \{u_1, u_2, ..., u_m\}$ in $K_{m,n}$. Then $G = K_{2m,n}$.

Since $K_{m,n}$, $\forall\ m, n \in N$ is absolute mean graceful graph proved by Kaneria and Chudasama [2], $K_{2m,n}$ is absolute mean graceful graph. Hence, G is absolute mean graceful graph. It also satisfies α - labeling. So, G is α - absolute mean graceful graph. □

Theorem 2.3 : Duplication of any vertex of $K_{m,n}$ by an edge is absolute mean graceful graph, but not α - absolute mean graceful.

***Proof*:** Let G be a graph obtained by duplication of vertex u_t, $(1 \le t \le m)$ of m-part of $K_{m,n}$ by an edge $u'_t u''_t$.

i.e. $V(G) = \{u_1, u_2, ..., u_m, v_1, v_2, ..., v_n\} \cup \{u'_t, u''_t\}$ and

$E(G) = \{\ u_i v_j\ /\ 1 \le i \le m, 1 \le j \le n\}\ \cup\ \{\ u_t u'_t, u_t u''_t, u'_t u''_t\}$.

i.e. $p = |V(G)| = m + n + 2$ and $q = mn + 3$.

Define $f : V(G) \to \{0,\ \pm 1,\ \pm 2,\ ...,\ \pm q\}$ as follows:

$$f(u_i) = \begin{cases} q, & i = 1 \\ f(u_{i-1}) - 2n, & i = 2,\ 3,\ ...,\ m. \end{cases}$$

$$f(v_j) = \begin{cases} -q, & i = 1 \\ f(v_{j-1}) +\ 2, & i = 2,\ 3,\ ...,\ n. \end{cases}$$

Case I : If $f(u_t) \ge 0$, then $f(u'_t) = f(u_t) - 5$ and $f(u''_t) = f(u_t) - 3$.
Case II : If $f(u_t) < 0$, then $f(u'_t) = f(u_t) + 5$ and $f(u''_t) = f(u_t) + 3$.

Similarly, it can be easily proved for duplication of vertex v_t $(1 \le t \le n)$ of n-part of $K_{m,n}$ by an edge $v'_t v''_t$ by replacing u_t, u'_t, u''_t as v_t, v'_t, v''_t respectively. It is clear that edge labeling function $f^* : E(G) \to \{1, 2, ..., q\}$ defined as $f^*(e) = \left\lceil \frac{|\ f(u) - f(v)\ |}{2} \right\rceil$ is bijective, for every edge $e = (u, v) \in E(G)$. Therefore, G is absolute mean graceful graph, but it is not α - absolute mean graceful graph. □

Theorem 2.4 : Duplication of any edge of $K_{m,n}$ by a new vertex is α - absolute mean graceful graph.

***Proof*:** Let G be a graph obtained by duplication of an edge $u_k v_l$ by a vertex w in $K_{m,n}$. So that $V(G) = V(K_{m,n}) \cup \{w\}$ and $E(G) = E(K_{m,n}) \cup$

$\{u_k w, wv_l\}$.i.e. $p = |V(G)| = m + n + 1$ and $q = mn + 2$. Define $f : V(G) \rightarrow \{0, \pm1, \pm2, ..., \pm q\}$ as follows:

$$f(u_i) = \begin{cases} q - 2ni, & i < k \\ q, & i = k \\ q - 2n(i-1), & i > k. \end{cases}$$

$$f(v_j) = \begin{cases} 2(i+1) - q, & j < l \\ 2 - q, & j = l \\ 2i - q, & j > l. \end{cases}$$

$f(w) = 1 - q$. It is clear that edge labeling function $f^* : E(G) \rightarrow \{1, 2, ..., q\}$ defined as $f^*(e) = \left\lceil \frac{| f(u) - f(v) |}{2} \right\rceil$ is bijective, for every edge $e = (u, v) \in E(G)$. Therefore, G is absolute mean graceful graph. It also satisfies α - labeling. So, G is α - absolute mean graceful graph. □

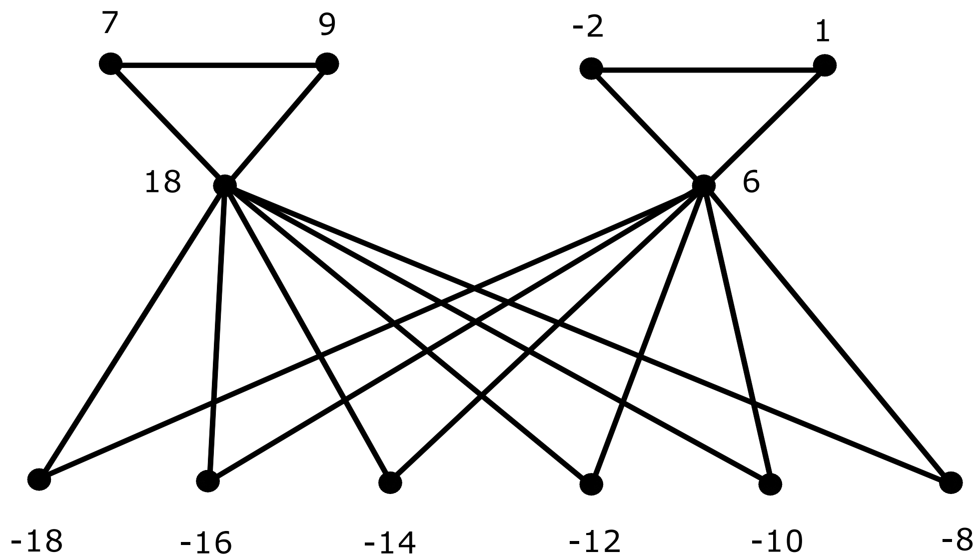
Theorem 2.5 : The graph obtained by duplication of both the vertices u_1, u_2 from 2-part in $K_{2,n}$, $\forall n \in N - \{2\}$ by edges is absolute mean graceful, but it is not α - absolute mean graceful graph.

***Proof*:** Let G be a graph obtained by duplication of both the vertices u_1, u_2 from 2-part in $K_{2,n}$, $\forall n \in N - \{2\}$ by edges $e_1 = u_1^{'} u_1^{''}$ and $e_2 = u_2^{'} u_2^{''}$.
i.e. $V(G) = V(K_{2,n}) \cup \{u_1^{'}, u_1^{''}, u_2^{'}, u_2^{''}\}$ and
$E(G) = E(K_{2,n}) \cup \{u_1 u_1^{'}, u_1^{'} u_1^{''}, u_1^{''} u_1, u_2 u_2^{'}, u_2^{'} u_2^{''}, u_2^{''} u_2\}$. i.e. $p = |V(G)| = n + 6$ and $q = 2n + 6$.

Define $f : V(G) \rightarrow \{0, \pm1, \pm2, ..., \pm q\}$ as follows:
$f(u_1^{'}) = q - 11$, $f(u_1^{''}) = q - 9$, $f(u_2^{'}) = -2$, $f(u_2^{''}) = 1$, $f(u_1) = q$, $f(u_2) = q - 2n$ $f(v_i) = -q + 2(i - 1)$, $\forall\, i = 1, 2, ..., n$.

It is clear that edge labeling function $f^* : E(G) \rightarrow \{1, 2, ..., q\}$ defined as $f^*(e) = \left\lceil \frac{| f(u) - f(v) |}{2} \right\rceil$ is bijective, for every edge $e = (u, v) \in E(G)$. Therefore, G is absolute mean graceful graph. It does not satisfy α-labeling. So, G is absolute mean graceful graph, but not α - absolute mean graceful graph. □

Illustration 1: Duplication of both the vertices from 2-part of $K_{2,6}$ by edges is absolute mean graceful graph. It is clear that $p = 12$ and $q = 18$.



Theorem 2.6 : Duplication of any vertex in cycle C_n, where $n \equiv 0\ (mod\, 2)$ is α - absolute mean graceful graph.

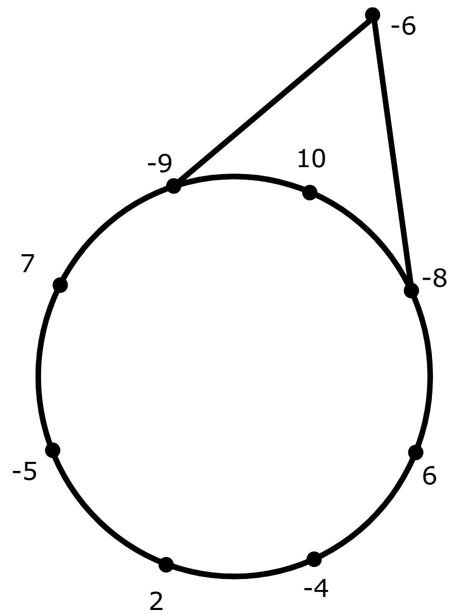
***Proof*:** Without loss of generality, let us assume that G is a graph obtained by duplication of vertex v_1 of cycle C_n, where $n \equiv 0\ (mod\, 2)$ by a new vertex u such that $N(v_1) = N(u)$. Assign the other vertices v_2, v_3,..., v_n of G in anticlockwise direction such that $V(G) = V(C_n) \cup \{u\}$ and $E(G) = E(C_n) \cup \{uv_n,\ uv_2\}$. Observe that $p = n+1$ and $q = n+2$

Define $f : V(G) \to \{0,\ \pm 1,\ \pm 2,\ ...,\ \pm q\}$ as follows:

$$f(v_i) = \begin{cases} q, & i = 1 \\ (-1)^{i+1}[\,|\,f(v_{i-1})\,|\,-2\,], & i = 2,\ 3, ...,\ \frac{q}{2}+1 \\ (-1)^{i+1}5, & i = \frac{q}{2}+2 \\ (-1)^{i+1}[\,|\,f(v_{i-1})\,|\,+2\,], & i = \frac{q}{2}+3,\ \frac{q}{2}+4, ...,\ n. \end{cases}$$

$f(u) = 4 - q$. It is clear that edge labeling function $f^* : E(G) \to \{1, 2, ..., q\}$ defined as $f^*(e) = \left\lceil \frac{|\,f(u) - f(v)\,|}{2} \right\rceil$ is bijective, for every edge $e = (u, v) \in E(G)$. Therefore, G is absolute mean graceful graph. It also satisfies α-labeling. So, G is α - absolute mean graceful graph. □

Illustration 2: Duplication of any vertex of cycle C_8 by a new vertex is α - absolute mean graceful graph. It is clear that $p = 9$ and $q = 10$.



Theorem 2.7 : Duplication of any vertex in path P_n, $\forall\, n \in N$ is α - absolute mean graceful graph.

***Proof*:** Let G be a graph obtained by duplication of any vertex v_k $(1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil)$ of path P_n by vertex u such that $V(G) = V(P_n) \cup \{u\}$ and

$$E(G) = \begin{cases} E(P_n) \cup \{v_2 u \ \ or \ \ v_{n-1} u\}, & if \ \ k = 1 \ \ or \ \ n \\ E(P_n) \cup \{v_{k-1} u, \ \ v_{k+1} u\}, & otherwise. \end{cases}$$

So that $p = n + 1$ and $q = n$ or $n + 1$. Define $f : V(G) \rightarrow \{0, \pm 1, \pm 2, \ldots, \pm q\}$ as follows:

$$f(v_i) = \begin{cases} q, & i = 1 \\ (-1)^{i+1}[\,|\, f(v_{i-1}) \,|\, -1\,], & i = 2, \ 3, \ldots, \ k + 2 \\ -f(v_{i-1}), & i = k + 3 \\ (-1)^{i+1}[\,|\, f(v_{i-1}) \,|\, -2\,], & i = k + 4 \\ (-1)^{i+1}[\,|\, f(v_{i-1}) \,|\, -1\,], & i = k + 5, \ k + 6, \ldots, \ n. \end{cases}$$

$$f(u) = \begin{cases} -q, & k = 1 \\ (-1)^k[\,|\,f(v_{k+1})\,|\,-2\,], & k = 2,\ 3, ...,\ \left\lceil \frac{n}{2} \right\rceil. \end{cases}$$

It is clear that edge labeling function $f^* : E(G) \to \{1, 2, ..., q\}$ defined as $f^*(e) = \left\lceil \frac{|\,f(u) - f(v)\,|}{2} \right\rceil$ is bijective, for every edge $e = (u, v) \in E(G)$. Therefore, G is absolute mean graceful graph. It also satisfies α-labeling. So, G is α - absolute mean graceful graph. □

Theorem 2.8 : Every swastik graph Sw_n, $\forall\, n \geq 4$ is α - absolute mean graceful graph.

***Proof*:** Let $v_{i,j}$ ($\forall\, i = 1,\ 2,\ 3,\ 4;\ \ \forall\, j = 1,\ 2,\ ...,\ 4n$) be vertices of i^{th} copy of C^i_{4n}. By combining vertices $v_{1,4t}$ and $v_{2,1}$; $v_{2,4t}$ and $v_{3,1}$; $v_{3,4t}$ and $v_{4,1}$; $v_{4,4t}$ and $v_{1,1}$ as four vertices, we can get the Swastik graph by bending and shaping. It is clear that $p = 16(n) - 4$ and $q = 16(n)$.

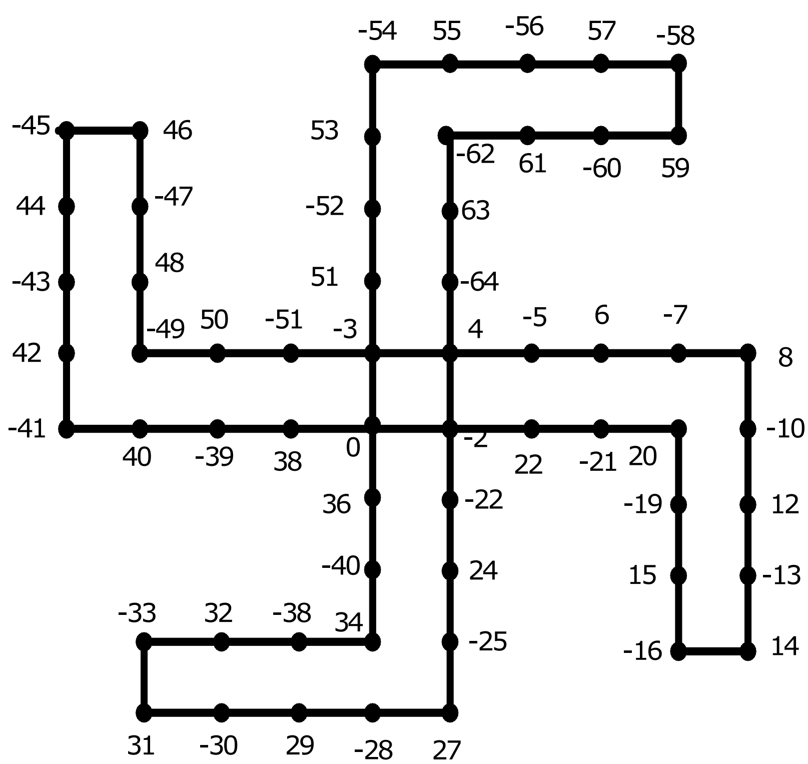
Define $f : V(Sw_n) \to \{0,\ \pm 1,\ \pm 2,\ ...,\ \pm q\}$ as follows:

$$f(v_{i,j}) = \begin{cases} 4, & i = 1\ \mathit{and}\ j = 1 \\ -q, & i = 1\ \mathit{and}\ j = 2 \\ (-1)^{j+1}[\,|\,f(v_{i,j-1})\,|\,-1\,], & i = 1\ \mathit{and}\ j = 3,\ 4, ...,\ 4n - 1 \\ -3, & i = 1\ \mathit{and}\ j = 4n \\ & \\ -(q - 4n + 3), & i = 2\ \mathit{and}\ j = 2 \\ (-1)^{j+1}[\,|\,f(v_{i,j-1})\,|\,-1\,], & i = 2\ \mathit{and}\ j = 3,\ 4, ...,\ 4n - 1 \\ 0, & i = 2\ \mathit{and}\ j = 4n \\ & \\ \frac{q}{2} + 4, & i = 3\ \mathit{and}\ j = 2 \\ -[\,f(v_{i,j-1}) + 4\,], & i = 3\ \mathit{and}\ j = 3,\ 5 \\ |\,f(v_{i,j-1})\,|\,-6, & i = 3\ \mathit{and}\ j = 4,\ 6 \\ -[\,|\,f(v_{i,j-1}) + 1\,], & i = 3\ \mathit{and}\ j = 7 \\ (-1)^j[\,|\,f(v_{i,j-1})\,|\,-2\,], & i = 3\ \mathit{and}\ j = 8,\ 2n + 5,\ 2n + 7 \\ (-1)^j[\,|\,f(v_{i,j-1})\,|\,-1\,], & i = 3\ \mathit{and}\ j = 9,\ 10, ...,\ 4n - 1; \\ & j \neq 2n + 5,\ 2n + 7 \\ (-2), & i = 3\ \mathit{and}\ j = 4n \end{cases}$$

$$f(v_{i,j}) = \begin{cases} \frac{q}{4} + 6, & i = 4 \ and \ j = 2 \\ (-1)^j [\,|\, f(v_{i,j-1}) \,|\, -1\,], & i = 4 \ and \ j = 3, \ 4, ..., \ 4n-1; \\ & j \neq 6, \ 7, \ 8, \ 2n+3, \ 2n+4 \\ [\, f(v_{i,j-1}) - 4\,], & i = 4 \ and \ j = 6 \\ -[\,|\, f(v_{i,j-1}) + 1\,], & i = 4 \ and \ j = 7 \\ [\,|\, f(v_{i,j-1}) - 2\,], & i = 4 \ and \ j = 8, \ 2n+3, \ 2n+4. \end{cases}$$

It is clear that induced edge labeling function $f^* : E(G) \to \{1, 2, ..., q\}$ defined as $f^*(e) = \left\lceil \frac{|\, f(u) - f(v) \,|}{2} \right\rceil$ is bijective, for every edge $e = (u, v) \in E(G)$. Therefore, G is absolute mean graceful graph. It also satisfies α - labeling. So, G is α - absolute mean graceful graph. □

Illustration 3: α - absolute mean graceful labeling in Sw_4. It is clear that $p = 60$ and $q = 64$.



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