# Some Results on Sequential Fractional Integro-Differential Equations with Anti-periodic Boundary Conditions 

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#### Abstract

In this paper, we study the existence of solutions for sequential fractional integrodifferential equations involving nonlocal anti-periodic boundary conditions. By using the fixed point theorems, the existence results are proved. An example is also support the main results.


Keywords: Existence; Fractional Derivative; anti-periodic boundary conditions; fixed point theorem.

## 1 Introduction

The boundary value problem is an valuable field in various research problems and the anti-periodic boundary conditions can be used in a different cases of applied problems. Infact, when anti-periodic boundary conditions are used instead of periodic boundary conditions, the numerical problems converge faster. However, the concept of parametric (nonlocal) anti-periodic boundary condition has not been addressed yet (see [2, 2,-5, 7,-17]) and reference therein.

In [6] authors studied a new concept of nonlocal anti-periodic boundary conditions involve a nonlocal intermediate point $0<a<T$ and the right end point $(t=T)$. Nonlocal anti-periodic boundary value problem of
(i) Nonlinear fractional differential equations(FDE's) of the form

$$
\begin{aligned}
& { }^{c} D^{\alpha} x(t)=f(t, x(t)), t \in[0, T], T>0,1<q \leq 2, \\
& x(a)=-x(T), x^{\prime}(a)=-x^{\prime}(T), 0<a<T,
\end{aligned}
$$

(ii) Nonlinear sequential FDE's of the form

$$
\begin{aligned}
& \left({ }^{c} D^{\alpha}+k^{c} D^{\alpha-1}\right) x(t)=f(t, x(t)), 1<\alpha \leq 2,0<t<T, \\
& x(a)=-x(T), x^{\prime}(a)=-x^{\prime}(T), 0<a \ll T,
\end{aligned}
$$

Motivated by the above work, consider a nonlinear sequential fractional integro differential equations with nonlocal anti-periodic boundary condition of the form

$$
\begin{align*}
& \left({ }^{c} D^{\vartheta}+k^{c} D^{\vartheta-1}\right) x(t)=f(t, x(t),(\chi x)(t)), 0<t<T,  \tag{1}\\
& x(a)=-x(T), x^{\prime}(a)=-x^{\prime}(T), 0<a \ll T, \tag{2}
\end{align*}
$$

where ${ }^{c} D^{\vartheta}$ denotes the Caputo fractional derivative of order $\vartheta, 1<\vartheta \leq 2, f:[0, T] \times \mathbb{R} \times$ $\mathbb{R} \rightarrow \mathbb{R}$ is given function, $g:[0, T] \times[0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is jointly continuous.

$$
(\chi x)(t)=\int_{0}^{t} g(t, s, x(s)) d s
$$

In this paper is planned as shades. In section 2, has definitions and elementary results of the fractional calculus. In section 3, the existence and uniqueness results for sequential fractional integro differential equations involving nonlocal anti-periodic boundary conditions are proved by using the standard fixed point theorems. In section 4, Some examples are illustrating the main results.

## 2 Preliminaries

Let us recall some basic definitions of fractional calculus. Let $\mathscr{P}=C([0, T], \mathbb{R})$ denote the Banach space of all continuous functions from $[0, T]$ into $\mathbb{R}$ endowed with the usual norm defined by $\|x\|=\sup \{|x(t)|, t \in[0, T]\}$.

Definition 1. The derivative of fractional order $\vartheta>0$ of a function $x:(0, \infty) \rightarrow \mathbb{R}$ is given by

$$
D_{0+}^{\vartheta} x(t)=\frac{1}{\Gamma(n-\vartheta)}\left(\frac{d}{d t}\right)^{n} \int_{0}^{t} \frac{x(s)}{(t-s)^{\vartheta-n+1}} d s
$$

where $n=[\vartheta]+1$, provided the right side is pointwise defined on $(0, \infty)$.
Definition 2. The fractional order integral of the function $f \in L^{1}\left([0, T], \mathbb{R}_{+}\right)$of order $\vartheta \in \mathbb{R}_{+}$is defined by

$$
I^{\vartheta} f(t)=\frac{1}{\Gamma(\vartheta)} \int_{0}^{t}(t-s)^{\vartheta-1} f(s) d s
$$

where $\Gamma$ is the Euler's gamma function defined by $\Gamma(\vartheta)=\int_{0}^{\infty} t^{\vartheta-1} e^{-t} d t, \vartheta>0$

Definition 3. for a continuous function $f:[0, \infty) \rightarrow \mathbb{R}$, the Caputo derivative of fractional order $\vartheta$ is defined as

$$
{ }^{c} D^{\vartheta} f(t)=\frac{1}{\Gamma(n-\vartheta)} \int_{0}^{t}(t-s)^{n-\vartheta-1} f^{n}(s) d s, n-1<\vartheta<n, n=[\vartheta]+1,
$$

provided that $f^{(n)}(t)$ exists, where $[\vartheta]$ denotes the integer part of the real number $\vartheta$
Lemma 1. Let $0<a \ll T$, and $h \in C([0, T], \mathbb{R})$. Then the unique solution of the problem:

$$
\begin{align*}
& \left({ }^{c} D^{\vartheta}+k^{c} D^{\vartheta-1}\right) x(t)=h(t), 1<\vartheta \leq 2,0<t<T, k \in \mathbb{R}^{+} \\
& x(a)=-x(T), x^{\prime}(a)=-x^{\prime}(T), 0<a \ll T, \tag{3}
\end{align*}
$$

is given by

$$
\begin{align*}
x(t)= & \int_{0}^{t} e^{-k(t-s)}\left(\int_{0}^{s} \frac{(s-p)^{\vartheta-2}}{\Gamma(\vartheta-1)} h(p) d p\right) d s \\
& +v_{1}(t)\left[\int_{0}^{T} \frac{(T-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} h(s) d s+\int_{0}^{a} \frac{(a-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} h(s) d s\right] \\
& +v_{2}(t)\left[\int_{0}^{T} e^{-k(T-s)}\left(\int_{0}^{s} \frac{(s-p)^{\vartheta-2}}{\Gamma(\vartheta-1)} h(p) d p\right) d s\right. \\
& \left.+\int_{0}^{a} e^{-k(a-s)}\left(\int_{0}^{s} \frac{(s-p)^{\vartheta-2}}{\Gamma(\vartheta-1)} h(p) d p\right) d s\right] \tag{4}
\end{align*}
$$

where

$$
\begin{gathered}
v_{1}(t)=\frac{2 e^{-k t}-\left(e^{-k T}+e^{-k a}\right)}{2 k\left(e^{-k T}+e^{-k a}\right)} \\
v_{2}(t)=\frac{-e^{-k t}}{e^{-k T}+e^{-k a}}
\end{gathered}
$$

Theorem 1. (Krasnoselkii's fixed point theorem) Let $K$ be a closed convex and nonempty subset of a banach space X.Let $T$ and $S$, be two operators such that
(i) $T x+S y \in K$ for any $x, y \in K$
(ii) $T$ is compact and continuous.
(iii) $S$ is contraction mapping.

Then there exists $z_{1} \in K$ such that $z_{1}=T z_{1}+S z_{1}$.

## 3 Main results

To prove the existence and uniqueness results we need the following assumptions :

- ( $A_{1}$ ) $\quad f:[0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.
- $\left(A_{2}\right)$ There exists constant $l>0$ such that

$$
\left|f(t, u, v)-f\left(t, u_{1}, v_{1}\right)\right| \leq l\left[\left|u-u_{1}\right|+\left|v-v_{1}\right|\right], \quad u, v, u_{1}, v_{1} \in \mathbb{R}
$$

for each $t \in J$, and each $u, u_{1}, v, v_{1} \in \mathbb{R}$

- $\left(A_{3}\right) \quad g:[0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and there exist a constant $l_{1}>0$, such that

$$
|g(t, s, x)-g(t, s, y)| \leq l_{1}|x-y|, x, y \in \mathbb{R}
$$

- $\left(A_{4}\right)$ there exists a function $\mu \in L^{1}([0, T], \mathbb{R})$ such that

$$
\sup |f(t, x,(\chi x))| \leq \mu(t), \forall(t, x,(\chi x)) \in[0, T] \times \mathbb{R} \times \mathbb{R}
$$

Theorem 2. Assume that the assumption $\left(A_{1}\right),\left(A_{2}\right)$ and $\left(A_{3}\right)$ are hold. If $l\left(1+l_{1}\right) A<1$, where

$$
\begin{equation*}
A=\frac{1}{k \Gamma(\vartheta)}\left[T^{\vartheta-1}\left(1-e^{-k T}\right)\left(1+\overline{v_{2}}\right)+k \overline{v_{1}}\left(T^{\vartheta-1}+a^{\vartheta-1}\right)+\overline{v_{2}} a^{\vartheta-1}\left(1-e^{-k a}\right)\right] \tag{5}
\end{equation*}
$$

then there exists a unique solution for the problem(1)-(2) on $[0, T]$

## Proof:

We transform the problem (1)-(2) into a fixed point problem and define the operator $\mathscr{M}$ : $C([0, T], \mathbb{R}) \rightarrow C([0, T], \mathbb{R})$ is given by

$$
\begin{align*}
\mathscr{M}(x)(t)= & \int_{0}^{t} e^{-k(t-s)}\left(\int_{0}^{s} \frac{(s-p)^{\vartheta-2}}{\Gamma(\vartheta-1)} f(p, x(p),(\chi x)(p)) d p\right) d s \\
& +v_{1}(t)\left[\int_{0}^{T} \frac{(T-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} f(s, x(s),(\chi x)(s)) d s+\int_{0}^{a} \frac{(a-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} f(s, x(s),(\chi x)(s)) d s\right] \\
& +v_{2}(t)\left[\int_{0}^{T} e^{-k(T-s)}\left(\int_{0}^{s} \frac{(s-p)^{\vartheta-2}}{\Gamma(\vartheta-1)} f(p, x(p),(\chi x)(p)) d p\right) d s\right. \\
& \left.+\int_{0}^{a} e^{-k(a-s)}\left(\int_{0}^{s} \frac{(s-p)^{\vartheta-2}}{\Gamma(\vartheta-1)} f(p, x(p),(\chi x)(p)) d p\right) d s\right] \tag{6}
\end{align*}
$$

where

$$
v_{1}(t)=\frac{2 e^{-k t}-\left(e^{-k T}+e^{-k a}\right)}{2 k\left(e^{-k T}+e^{-k a}\right)}, \quad v_{2}(t)=\frac{-e^{-k t}}{e^{-k T}+e^{-k a}} .
$$

It can easily be shown that $\mathscr{M} B_{r} \subset B_{r}$, where $B_{r}=\{x \in C([0, T], \mathbb{R}:\|x\| \leq r\}$
Next we show that the operator $\mathscr{M}$ is a contraction. for $x, y \in C([0, T], \mathbb{R})$ and $t \in[0, T]$, we have

$$
\begin{aligned}
\mid & \mathscr{M}(x)-\mathscr{M}(y) \mid \\
\leq & \max _{t \in[0, T]} \left\lvert\, \int_{0}^{t} e^{-k(t-s)}\left(\int_{0}^{s} \frac{(s-p)^{\vartheta-2}}{\Gamma(\vartheta-1)}(f(p, x(p),(\chi x)(p))-f(p, y(p),(\chi y)(p))) d p\right) d s\right. \\
& +v_{1}(t)\left[\int_{0}^{T} \frac{(T-s)^{\vartheta-2}}{\Gamma(\vartheta-1)}(f(s, x(s),(\chi x)(s))-f(s, y(s),(\chi y)(s))) d s\right. \\
& \left.+\int_{0}^{a} \frac{(a-s)^{\vartheta-2}}{\Gamma(\vartheta-1)}(f(s, x(s),(\chi x)(s))-f(s, y(s),(\chi y)(s))) d s\right] \\
& +v_{2}(t)\left[\int_{0}^{T} e^{-k(T-s)}\left(\int_{0}^{s} \frac{(s-p)^{\vartheta-2}}{\Gamma(\vartheta-1)}(f(p, x(p),(\chi x)(p))-f(p, y(p),(\chi y)(p))) d p\right) d s\right. \\
& \left.+\int_{0}^{a} e^{-k(a-s)}\left(\int_{0}^{s} \frac{(s-p)^{\vartheta-2}}{\Gamma(\vartheta-1)}(f(p, x(p),(\chi x)(p))-f(p, y(p),(\chi y)(p))) d p\right) d s\right] \mid \\
\leq & \frac{l\left(1+l_{1}\right)}{k \Gamma(\vartheta)}\left[T^{\vartheta-1}\left(1-e^{-k T}\right)\left(1+\overline{v_{2}}\right)+k \overline{v_{1}}\left(T^{\vartheta-1}+a^{\vartheta-1}\right)+\overline{v_{2}} a^{\vartheta-1}\left(1-e^{-k a}\right)\right]\|x-y\|
\end{aligned}
$$

By (5), the operator $\mathscr{M}$ is a continuous. Hence by Banach's contraction principle, $\mathscr{M}$ has a unique fixed point which is a unique solution of the problem (1) - (2).

Theorem 3. Assume that $\left(A_{1}\right) \&\left(A_{4}\right)$ are holds. The problem (1)-(2) has at least one solution on J provided that

$$
\begin{equation*}
\frac{1}{k \Gamma(\vartheta)}\left[T^{\vartheta-1}\left(1-e^{-k T}\right)\left(1+\overline{v_{2}}\right)+k \overline{v_{1}}\left(T^{\vartheta-1}+a^{\vartheta-1}\right)+\overline{v_{2}} a^{\vartheta-1}\left(1-e^{-k a}\right)\right]<1 \tag{7}
\end{equation*}
$$

## Proof:

Let us define $B_{r}=\{x \in C([0, T]), \mathbb{R}:\|x\| \leq r\}$ with $r \geq 2\left(\frac{\left(1-e^{-T k}\right)}{k} T^{\vartheta-1}+V_{1}\left(T^{\vartheta-1}-a^{\vartheta-1}\right)+V_{2}\left[\frac{\left(1-e^{-T k}\right)}{k} T^{\vartheta-1}+\frac{\left(1-e^{-a k}\right)}{k} a^{\vartheta-1}\right]\right)$
and consider the operators $\mathscr{M}_{1}$ and $\mathscr{M}_{1}$ on $B_{r}$ as

$$
\begin{aligned}
\left(\mathscr{M}_{1} x\right)(t)= & \int_{0}^{t} e^{-k(t-s)}\left(\int_{0}^{s} \frac{(s-p)^{\vartheta-2}}{\Gamma(\vartheta-1)} f(p, x(p),(\chi x)(p)) d p\right) d s \\
\left(\mathscr{M}_{2} x\right)(t)= & v_{1}(t)\left[\int_{0}^{T} \frac{(T-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} f(s, x(s),(\chi x)(s)) d s\right. \\
& \left.+\int_{0}^{a} \frac{(a-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} f(s, x(s),(\chi x)(s)) d s\right] \\
& +v_{2}(t)\left[\int_{0}^{T} e^{-k(T-s)}\left(\int_{0}^{s} \frac{(s-p)^{\vartheta-2}}{\Gamma(\vartheta-1)} f(p, x(p),(\chi x)(p)) d p\right) d s\right. \\
& \left.+\int_{0}^{a} e^{-k(a-s)}\left(\int_{0}^{s} \frac{(s-p)^{\vartheta-2}}{\Gamma(\vartheta-1)} f(p, x(p),(\chi x)(p)) d p\right) d s\right]
\end{aligned}
$$

For $x, y \in B_{r}$, we find that

$$
\begin{aligned}
\left\|\mathscr{M}_{1} x+\mathscr{M}_{2} y\right\| \leq & \frac{\left(1-e^{-T k}\right)}{k}\left(\frac{\|\mu\|_{L^{1}}}{\Gamma(\vartheta)}\right) T^{\vartheta-1} \\
& +V_{1}\left(\frac{\|\mu\|_{L^{1}}}{\Gamma(\vartheta)}\right)\left(T^{\vartheta-1}-a^{\vartheta-1}\right) \\
& +V_{2}\left[\frac{\left(1-e^{-T k}\right)}{k} T^{\vartheta-1}+\frac{\left(1-e^{-a k}\right)}{k} a^{\vartheta-1}\right]\left(\frac{\|\mu\|_{L^{1}}}{\Gamma(\vartheta)}\right) \\
\leq & \left(\frac{\left(1-e^{-T k}\right)}{k} T^{\vartheta-1}+V_{1}\left(T^{\vartheta-1}-a^{\vartheta-1}\right)\right. \\
& \left.+V_{2}\left[\frac{\left(1-e^{-T k}\right)}{k} T^{\vartheta-1}+\frac{\left(1-e^{-a k}\right)}{k} a^{\vartheta-1}\right]\right)\left(\frac{\|\mu\|_{L^{1}}}{\Gamma(\vartheta)}\right) \\
\left\|\mathscr{M}_{1} x+\mathscr{M}_{2} y\right\| \leq & r
\end{aligned}
$$

Thus, $\mathscr{M}_{1} x+\mathscr{M}_{2} y \in B_{r}$. It follows from the assumption $\left(A_{4}\right)$ that $\mathscr{M}_{2}$ is a contraction mapping. $\mathscr{M}_{1}$ is a continuous. Also $\mathscr{M}_{1}$ is uniformly bounded on $B_{r}$ as

$$
\left\|\mathscr{M}_{1}(x)\right\| \leq \frac{\left(1-e^{-T k}\right)}{k \Gamma(\vartheta)} T^{\vartheta-1}\|\mu\|_{L^{1}}
$$

Now let us prove that $\left(\mathscr{M}_{1} x\right)(t)$ is equicontinuous. Let $t_{1}, t_{2} \in[0, T], t_{1}<t_{2}$ and $x \in B_{r}$. Using the fact that $f$ is bounded on the compact set $[0, T] \times B_{r} \times \chi\left(B_{r}\right)$, we define

$$
\sup _{(t, x, y) \in[0, T] \times B_{r} \times \chi\left(B_{r}\right)}\|f(t, x(s),(\chi x)(s))\|=C_{0}<\infty
$$

consequently, we have

$$
\begin{aligned}
\left\|\left(\mathscr{M}_{1} x\right)\left(t_{2}\right)-\left(\mathscr{M}_{1} x\right)\left(t_{1}\right)\right\| \leq & \left|e^{-k t_{2}}-e^{-k t_{1}}\right| \int_{0}^{t_{1}} e^{k s}\left(C_{0} \int_{0}^{s} \frac{(s-p)^{\vartheta-2}}{\Gamma(\vartheta-1)} d p\right) d s \\
& +\int_{t_{1}}^{t_{2}} e^{-k\left(t_{2}-s\right)}\left(C_{0} \int_{0}^{s} \frac{(s-p)^{\vartheta-2}}{\Gamma(\vartheta-1)} d p\right) d s
\end{aligned}
$$

which is independent of $x$ and tends to zero as $t_{2} \rightarrow t_{1}$. Thus $\mathscr{M}_{1}\left(B_{r}\right)$ is relatively compact. Hence by the Ascoli-Arzela theorem, $\mathscr{M}_{1}$ is compact on $B_{r}$. Thus all the assumption of theorem 1 are satisfied and hence problem (1) - (2) has at least one solution on $[0, T]$.

## 4 Example

Let us consider the a nonlocal anti-periodic boundary value problem of nonlinear sequential fractional integro differential equations of the form

$$
\begin{equation*}
\left({ }^{c} D^{\frac{3}{2}}+5^{c} D^{\frac{1}{2}}\right) x(t)=\frac{|x|}{2\left(e^{t}+1\right)^{2}(1+|x|)}+\frac{1}{2} \int_{0}^{t} e^{-\frac{1}{4} x(s)} d s, \quad 1<\vartheta \leq 2, \quad t \in J:[0, T] \tag{8}
\end{equation*}
$$

$$
\begin{equation*}
x(0.01)=-x(1), x^{\prime}(0.01)=-x^{\prime}(1) \tag{9}
\end{equation*}
$$

Here $\vartheta=\frac{3}{2}, k=5, a=0.01, T=1$ and

$$
\begin{aligned}
f(t, x(t),(\chi x)(t)) & =\frac{|x|}{2\left(e^{t}+1\right)^{2}(1+|x|)}+(\chi x)(t) \\
(\chi x)(t) & =\frac{1}{2} \int_{0}^{t} e^{-\frac{1}{4} x(s)} d s
\end{aligned}
$$

Hence the conditions (A2)-(A3) are holds with

$$
\begin{gathered}
l=\frac{1}{8}, l_{1}=\frac{1}{2} \\
v_{1}=0.108775, v_{2}=1.043877, A=0.594308
\end{gathered}
$$

We shall check that condition, indeed

$$
\frac{l\left(1+l_{1}\right)}{k \Gamma(\vartheta)}\left[T^{\vartheta-1}\left(1-e^{-k T}\right)\left(1+\overline{v_{2}}\right)+k \overline{v_{1}}\left(T^{\vartheta-1}+a^{\vartheta-1}\right)+\overline{v_{2}} a^{\vartheta-1}\left(1-e^{-k a}\right)\right]<1
$$

which is satisfied for some $1<\vartheta \leq 2$. Then by Theorem 2 the problem (1)-(2) has a unique solution.

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