On The Negative Pell Equation $y^2 = 30x^2 - 45$

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ABSTRACT:

The binary quadratic Diophantine equation represented by the negative pellian $y^2 = 30x^2 - 45$ is analyzed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas.

KEYWORDS: *Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.*

I. INTRODUCTION

The binary quadratic equations of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been selected by various mathematicians for its non-trivial integer solutions when D takes different integral values[1-4].For an extensive review of various problems, one may refer[5-10]. In this communication, yet another interesting equation given by $y^2 = 30x^2 - 45$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

II. METHOD OF ANALYSIS

The negative Pell equation representing hyperbola under consideration is

$$y^2 = 30x^2 - 45 \tag{1}$$

whose smallest positive integer solution is

$$x_0 = 3,\ y_0 = 15.$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2 = 30x^2 + 1 \tag{2}$$

whose initial solution is given by

$$\tilde{x}_n = 2,\ \tilde{y}_n = 11$$

The general solution $(\tilde{x}_n, \tilde{y}_n)$ of (2) is given by,

$$\tilde{x}_n = \frac{1}{2\sqrt{30}} g_n,\ \tilde{y}_n = \frac{1}{11} f_n$$

where,

$$f_n = \left(11+2\sqrt{30}\right)^{n+1} + \left(11-2\sqrt{30}\right)^{n+1}$$

$$g_n = \left(11+2\sqrt{30}\right)^{n+1} - \left(11-2\sqrt{30}\right)^{n+1} \quad , \quad n=-1,0,1,2......$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$,the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{1}{2\sqrt{30}} g_n$$

$$y_{n+1} = f_n + \sqrt{30} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 22x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 22y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table: 1 below:

Table:1 Numerical examples

n	x_{n+1}	y_{n+1}
-1	1	2
0	3	15
1	63	345
2	1383	7575
3	30363	166305
4	666603	3651135

From the above table, we observe some interesting relations among the solutions which are presented below:

1. x_{n+1} is always odd, y_{n+1} is always odd.

2.Relations among the solutions

- $22\,x_{n+2} - x_{n+1} - x_{n+3} = 0$
- $2y_{n+1} - x_{n+2} + 11x_{n+1} = 0$
- $2y_{n+2} - 11x_{n+2} + x_{n+1} = 0$

- $2y_{n+3} - 241x_{n+2} + 11x_{n+1} = 0$
- $x_{n+3} - 241x_{n+3} - 44y_{n+1} = 0$
- $2399y_{n+1} - 14520x_{n+1} - y_{n+3} = 0$
- $120x_{n+1} - y_{n+1} - y_{n+2} = 0$
- $60x_{n+1} - 241y_{n+2} - 11y_{n+3} = 0$
- $28980x_{n+1} - 11y_{n+2} - 241y_{n+3} = 0$
- $11x_{n+3} - 241x_{n+2} - 2y_{n+1} = 0$
- $11x_{n+2} - x_{n+3} - 2y_{n+2} = 0$
- $11x_{n+3} - x_{n+2} - 2y_{n+3} = 0$
- $2y_{n+1} - x_{n+2} - 11x_{n+1} = 0$
- $60x_{n+2} - 11y_{n+2} + y_{n+1} = 0$
- $11y_{n+1} - 1320x_{n+2} - 11y_{n+3} = 0$
- $11y_{n+2} - 60x_{n+2} - y_{n+3} = 0$
- $11x_{n+3} - 241x_{n+3} + 2y_{n+1} = 0$
- $11y_{n+1} - 60x_{n+3} - 241y_{n+2} = 0$
- $1320x_{n+3} + y_{n+1} - 241y_{n+3} = 0$
- $2y_{n+2} + 11x_{n+2} - 11x_{n+3} = 0$

3. Each of the following expressions represents a nasty number

- $\frac{1}{3}\left(138x_{2n+2} - 6x_{2n+3} + 12\right)$
- $\frac{1}{66}\left(3030x_{2n+2} - 6x_{2n+4} + 12\right)$
- $\frac{2}{3}\left(36x_{2n+1} - 6y_{2n+2} + 12\right)$
- $\frac{2}{495}\left(11340x_{2n+2} - 90y_{2n+3} + 12\right)$

- $\frac{2}{723}\left(16596x_{2n+2}-6y_{2n+4}+12\right)$
- $\frac{2}{90}\left(45450x_{2n+3}-2070x_{2n+4}+12\right)$
- $\frac{2}{495}6\left(90x_{2n+3}-345y_{2n+2}\right)+2$
- $\frac{2}{45}6\left(1890x_{2n+4}-345y_{2n+4}\right)+2$
- $\frac{2}{495}6\left(41490x_{2n+3}-345y_{2n+4}\right)+2$
- $\frac{2}{10845}6\left(90x_{2n+4}-7575y_{2n+2}\right)+2$
- $\frac{2}{495}6\left(1890x_{2n+4}-7575y_{2n+3}\right)+2$
- $\frac{2}{45}\left(41490x_{2n+4}-7575y_{2n+4}\right)+2$
- $\frac{2}{2700}6\left(1890y_{2n+2}-90y_{2n+3}\right)+2$
- $\frac{2}{59400}6\left(41490y_{2n+2}-90y_{2n+4}\right)+2$
- $\frac{2}{2700}6\left(41490y_{2n+3}-1890y_{2n+4}\right)+2$

4. Each of the following expressions represents a cubical integer

- $\frac{1}{3}\left[23x_{3n+3}-x_{3n+4}+69x_{n+1}-3x_{n+2}\right]$
- $\frac{1}{66}\left[505x_{3n+3}-x_{3n+5}+1515x_{n+1}-3x_{n+3}\right]$
- $\frac{2}{3}\left[6x_{3n+3}-y_{3n+3}+18x_{n+1}-3y_{n+1}\right]$
- $\frac{2}{495}\left[1890x_{3n+3}-15y_{3n+4}+5670x_{n+1}-45y_{n+2}\right]$
- $\frac{2}{723}\left[2766x_{3n+3}-y_{3n+5}+8298x_{n+1}-3y_{n+3}\right]$

- $\frac{2}{90}\left[7575x_{3n+4}-345x_{3n+5}+22725x_{n+1}-1035x_{n+3}\right]$
- $\frac{2}{495}\left[90x_{3n+4}-345y_{3n+3}+270x_{n+2}-1035y_{n+1}\right]$
- $\frac{2}{45}\left[1890x_{3n+4}-345y_{3n+4}+5670x_{n+2}-1035y_{n+2}\right]$
- $\frac{2}{495}\left[41490x_{3n+4}-345y_{3n+5}+124470x_{n+2}-1035y_{n+3}\right]$
- $\frac{2}{10845}\left[90x_{3n+5}-7575y_{3n+3}+270x_{n+3}-22725y_{n+1}\right]$
- $\frac{2}{495}\left[1890x_{3n+5}-7575y_{3n+4}+5670x_{n+3}-227253y_{n+2}\right]$
- $\frac{2}{45}\left[41490x_{3n+5}-7575y_{3n+5}+124470x_{n+3}-22725y_{n+3}\right]$
- $\frac{2}{2700}\left[1890y_{3n+3}-90y_{3n+4}+5670y_{n+1}-270y_{n+2}\right]$
- $\frac{2}{59400}\left[41490y_{3n+3}-90y_{3n+5}+124470y_{n+1}-270y_{n+3}\right]$
- $\frac{2}{2700}\left[41490y_{3n+4}-1890y_{3n+5}+124470y_{n+2}-5670y_{n+3}\right]$

5. Each of the following expressions represents a bi-quadratic integer

- $\frac{1}{3}\left[23x_{4n+4}-x_{4n+5}+92x_{2n+2}-4x_{2n+3}+18\right]$
- $\frac{1}{66}\left[505x_{4n+4}-x_{4n+6}+2020x_{2n+2}-4x_{2n+4}+396\right]$
- $\frac{2}{3}\left[6x_{4n+4}-y_{4n+4}+24x_{2n+2}-4y_{2n+2}+18\right]$
- $\frac{2}{495}\left[1890x_{4n+4}-15y_{4n+5}+7560x_{2n+2}-60y_{2n+3}+2970\right]$
- $\frac{2}{723}\left[2766x_{4n+4}-y_{4n+6}+1064x_{2n+2}-4y_{2n+4}+4338\right]$
- $\frac{2}{90}\left[7575x_{4n+5}-345y_{4n+6}+30300x_{2n+3}-1380x_{2n+4}+540\right]$

- $\frac{2}{495}\left[20x_{4n+5}-345y_{4n+4}+360x_{2n+3}-1380y_{2n+2}+2970\right]$
- $\frac{2}{45}\left[1890x_{4n+5}-345y_{4n+5}+7560x_{n+3}-1380y_{n+3}+2970\right]$
- $\frac{2}{495}\left[41490x_{4n+5}-345y_{4n+6}+165960x_{2n+3}-1380y_{2n+4}+2970\right]$
- $\frac{2}{10845}\left[90x_{4n+6}-7575y_{4n+4}+360x_{2n+4}-30300y_{2n+2}+65070\right]$
- $\frac{2}{495}\left[1890x_{4n+6}-7575y_{4n+5}+7560x_{2n+4}-30300y_{2n+3}+2970\right]$
- $\frac{2}{45}\left[41490x_{4n+6}-7575y_{4n+6}+165960x_{2n+4}-30300y_{2n+4}+2970\right]$
- $\frac{2}{2700}\left[1890y_{4n+4}-90y_{4n+5}+7560y_{2n+3}-360y_{2n+3}+16200\right]$
- $\frac{2}{59400}\left[41490y_{4n+4}-90y_{4n+6}+165960y_{2n+2}-360y_{2n+4}+356400\right]$
- $\frac{2}{2700}\left[41490y_{4n+5}-1890y_{4n+6}+165960y_{2n+3}-7560y_{2n+4}+16200\right]$

6. Each of the following expressions represents a quintic integer

- $\frac{1}{3}\left[23x_{5n+5}-x_{5n+6}+115x_{3n+3}-5x_{3n+4}+460x_{n+1}-20x_{n+2}\right]$
- $\frac{1}{66}\left[505x_{5n+5}-x_{5n+7}+2525x_{3n+4}-5x_{3n+5}+10100x_{n+1}-20x_{n+3}\right]$
- $\frac{2}{3}\left[6x_{5n+5}-y_{5n+7}+30x_{3n+3}-5y_{3n+5}+120x_{n+1}-20y_{n+1}\right]$
- $\frac{2}{495}\left[1890x_{5n+5}-15y_{5n+6}+9450x_{3n+3}-75y_{3n+4}-37800x_{n+1}-300y_{n+1}\right]$
- $\frac{2}{723}\left[2766x_{5n+5}-y_{5n+7}+13830x_{3n+3}-5y_{5n+5}+55320x_{n+1}-20y_{n+2}\right]$
- $\frac{2}{90}\left[7575x_{5n+6}-345y_{5n+7}+37875x_{3n+4}-1725x_{3n+5}+151500x_{n+1}-6900y_{n+3}\right]$
- $\frac{2}{495}\left[90x_{5n+6}-345y_{5n+5}+490x_{3n+4}-1725y_{3n+5}+1800x_{n+2}-6900y_{n+2}\right]$

- $\frac{2}{45}\left[1890x_{5n+6}-345y_{5n+6}+9450x_{3n+4}-1725y_{3n+4}+37800x_{n+2}-3450y_{n+2}\right]$
- $\frac{2}{495}\left[41490x_{5n+6}-345y_{5n+7}+207450x_{3n+4}-1725y_{3n+5}+829800x_{n+2}-6900y_{n+3}\right]$
- $\frac{2}{10845}\left[90x_{5n+7}-7575y_{5n+5}+40x_{3n+5}-37875y_{3n+3}+1800x_{n+3}-151500y_{n+1}\right]$
- $\frac{2}{495}\left[1890x_{5n+7}-7575y_{5n+7}+9450x_{3n+5}-37875y_{3n+4}+37800x_{n+3}-1175490y_{n+2}\right]$
- $\frac{2}{45}\left[41490x_{5n+7}-7575y_{5n+7}+207450x_{3n+5}-37875y_{3n+5}+829800x_{n+3}-151500y_{n+3}\right]$
- $\frac{2}{2700}\left[1890y_{5n+5}-90y_{5n+6}+9450y_{3n+3}-450y_{3n+4}+37800y_{n+1}-1800y_{n+2}\right]$
- $\frac{2}{59400}\left[41490y_{5n+3}-90y_{5n+7}+450y_{3n+5}-207450y_{3n+3}+829800y_{n+1}-1800y_{n+3}\right]$
- $\frac{2}{2700}\left[41490y_{5n+6}-1890y_{5n+7}+207450y_{3n+5}-9450y_{3n+4}+829800y_{n+2}-37800y_{n+3}\right]$

REMARKABLE OBSERVATIONS:

1. Employing linear combinations among the solutions of (1),one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below:

Table: 2 Hyperbolas

S. No	Hyperbola	(X,Y)
1	$6X^2-5Y^2=180$	$(23x_{n+1}-x_{n+2},\ x_{n+2}-21x_{n+1})$
2	$6X^2-5Y^2=87120$	$(505x_{n+1}-x_{n+3},-461x_{n+1}+x_{n+3})$
3	$6X^2-5Y^2=45$	$(6x_{n+1}-y_{n+1},-5x_{n+1}+y_{n+1})$
4	$60X^2-5Y^2=490050$	$(1890x_{n+1}-15y_{n+2},3y_{n+2}-345x_{n+1})$
5	$24X^2-20Y^2=10454580$	$(2766x_{n+1}-y_{n+3},y_{n+3}-2525x_{n+1})$

6	$60X^2-2Y^2=16200$	$(7575x_{n+2}-345x_{n+3},-1383x_{n+2}-63x_{n+3})$
7	$60X^2-2Y^2=245025$	$(90x_{n+2}-345y_{n+1},63y_{n+1}-15x_{n+2})$
8	$60X^2-2Y^2=4050$	$(1890x_{n+2}-345y_{n+1},63y_{n+2}-345x_{n+2})$
9	$60X^2-2Y^2=490050$	$(41490x_{n+2}-345y_{n+3},63\,y_{n+3}-7575x_{n+2})$
10	$60X^2-2Y^2=235228050$	$(90x_{n+3}-7575y_{n+1},383\,y_{n+1}-15x_{n+3})$
11	$60X^2-2Y^2=490050$	$(1890x_{n+3}-7575y_{n+2},1383\,y_{n+2}-345x_{n+3})$
12	$60X^2-2Y^2=4050$	$(41490x_{n+3}-7575y_{n+3},1383y_{n+3}-7575x_{n+3})$
13	$60X^2-2Y^2=14580000$	$(-90y_{n+2}+1890y_{n+1},-345y_{n+1}-15y_{n+2})$
14	$60X^2-2Y^2=7056720000$	$(90y_{n+3}-41490y_{n+1},-7575y_{n+1}-15y_{n+3})$
15	$60X^2-2Y^2=14580000$	$(1890y_{n+3}-41490y_{n+2},7575y_{n+2}-345y_{n+3})$

2. Employing linear combinations among the solutions of (1),one may generate integer solutions for other choices of parabola which are presented in the Table: 3 below:

Table: 3 Parabolas

S. No	Parabola	(X,Y)
1	$2Y-5X^2=30$	$(23x_{2n+2}-x_{2n+3},x_{n+2}-21x_{n+1})$
2	$110Y-2X^2=14520$	$(505x_{2n+1}-x_{2n+3},461x_{n+3}-x_{n+3})$
3	$10Y-8X^2=30$	$(6x_{2n+1}-y_{2n+1},5\,x_{n+1}-y_{n+1})$
4	$66Y-8X^2=32670$	$(1890x_{2n+1}-15y_{2n+2},3y_{n+2}-345x_{n+1})$
5	$3615Y-12X^2=2613645$	$(2766x_{2n+1}-y_{2n+3},y_{n+3}-2525x_{n+1})$
6	$90Y-60X^2=8100$	$(7575x_{2n+2}-345x_{2n+3},1383x_{n+2}-63x_{n+3})$
7	$990Y-120X^2=490050$	$(90x_{2n+2}-345y_{2n+1},63y_{n+1}-15x_{n+2})$
8	$45Y-60X^2=2025$	$(41490x_{2n+1}-345y_{2n+2},63y_{n+2}-345x_{n+2})$

9	$198Y - 24X^2 = 98010$	$\left(41490x_{2n+1} - 345y_{2n+3}, 63y_{n+3} - 7575x_{n+2}\right)$
10	$10845Y - 60X^2 = 117614025$	$\left(90x_{2n+3} - 7575y_{2n+2} +, 1383y_{n+1} - 15x_{n+3}\right)$
11	$198Y - 24X^2 = 98010$	$\left(1890x_{2n+3} - 7575y_{2n+2}, 1383y_{n+2} - 345x_{n+3}\right)$
12	$18Y - 24X^2 = 810$	$\left(41490x_{2n+3} - 7575y_{2n+3}, 1383y_{n+3} - 7575x_{n+3}\right)$
13	$1080Y - 24X^2 = 2916000$	$\left(25y_{2n+4} - 949y_{2n+3} + 108, 3001y_{n+2} - 79y_{n+3}\right)$
14	$23760Y - 24X^2 = 141344000$	$\left(41490y_{2n+1} - 90y_{2n+3}, 7575y_{n+1} - 15y_{n+3}\right)$
15	$1080Y - 24X^2 = 2916000$	$\left(41490y_{2n+2} - 1890y_{2n+3}, 7575y_{n+2} - 345y_{n+3}\right)$

III. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the negative Pell Equation $y^2 = 30x^2 - 45$ .As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties.

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