Finite Difference Method For Solving Fuzzy Linear Boundary Value Problems

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ABSTRACT:

In this paper, the Finite Difference method is considered to solve the nonhomogeneous for Second order fuzzy boundary value problems, In which the fuzziness appeared together in the boundary conditions and in the nonhomogeneous term of the differential equation. The method of solution depends on transforming the fuzzy problem to equivalent crisp problems using the concept of α-level sets.

KEYWORDS: *the Finite Difference Method; Fuzzy sets; α-level sets; second order fuzzy boundary value problems.*

I. INTRODUCTION

A fuzzy set theory had been introduced by Zadeh in 1965, in which, Zadeh's original definition of fuzzy set is as follows: A fuzzy set is a class of objects with a continuum grades of membership, such a set is characterized by a membership or characteristic function which assigns to each object a grade of membership ranging between zero and one, [1].

Kandel [5] applied the concept of fuzzy differential equations to the analysis of fuzzy dynamical problems, but the boundary value problems was treated, rigorously by Lakshmikantham, V., Murty, K. N. and Turner, J. in 2001, [8]. Henderon, J. and Peterson, A. in 2004 obtained a theorem of the existence and uniqueness of solutions for the boundary value problems of fuzzy differential equations.

Pearson in 1997 [3], introduced the analytical method for solving linear system of fuzzy differential equations.

In this paper, the Finite Difference method is applied to solve the second order fuzzy boundary value problems. This method we will present here have better stability characteristics, but generally require more computations to obtain a pre-specified accuracy.

The purpose of this paper is involving finite differences for solving boundary value problems consists of replacing each of the derivatives in the differential equation by an appropriate difference approximation. The difference equation is generally chosen so that a certain order of truncation error is maintained [9]. Consider the following second order fuzzy boundary value problem

$$y'' = p(x)y' + q(x)y + r(x), \qquad a \le x \le b$$

$$y(a) \simeq \tilde{\alpha}, \quad y(b) \simeq \tilde{\beta}$$

The structure of the paper include the following: in section 2, we provide some important definitions and basic results to be used in this paper. in section 3 we present Finite Difference Method . In section 4, we illustrate proposed method by numerical example.

II. FUZZY SETS:

In this section, we shall present some basic definitions of fuzzy sets including the definition of fuzzy numbers and fuzzy functions.

Definition (1), [6]:

Let X be any set of elements. A fuzzy set $\tilde{A}$ is characterized by a membership function $\mu_{\tilde{A}}$: X$\to$ [0, 1], and may be written as the set of points

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X, 0 \le \mu_{\tilde{A}}(x) \le 1\}$$

Definition (2), [4]:

The crisp set of elements that belong to the set $\tilde{A}$ at least to the degree α is called the weak α -level set (or weak α -cut), and is defined by:

$$A_{\alpha} = \{x \in X: \mu_{\tilde{A}}(x) \ge \alpha\}$$

while the strong α -level set (or strong α -cut) is defined by:

$$A'_{\alpha} = \{x \in X: \mu_{\tilde{A}}(x) > \alpha\}$$

Definition (3),[7]:

A fuzzy subset $\tilde{A}$ of a universal space X is convex if and only if the sets A_α are convex, $\forall \alpha \in [0,1]$. Or equivalently, we can define convex fuzzy set directly by using its membership function to satisfy:

$$\mu_{\tilde{A}}[\lambda x_1 + (1-\lambda)x_2] \geq Min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$$

For all $x_1, x_2 \in X$ and $\lambda \in [0,1]$.

Definition (4),[2]:

A fuzzy number $\tilde{M}$ Is a convex normalized fuzzy set $\tilde{M}$ Of the real line R, such that:

1. There exists exactly one $x_0 \in R$, with $\mu_{\tilde{M}}(x_0) = 1$ (x_0 Is called the mean value of $\tilde{M}$).

2. $\mu_{\tilde{M}}(x)$ Is piecewise continuous

Now, in applications, the representation a fuzzy number in terms of its membership function is so difficult to use, therefore two approaches are given for representing the fuzzy number in terms of its α -level sets, as in the following remark:

Remark (1):

A fuzzy number $\tilde{M}$ may be uniquely represented in terms of its $\square$--level sets, as the following closed intervals of the real line:

$$M_\alpha = [m - \sqrt{1-\alpha}, m + \sqrt{1-\alpha}] \quad or M_\alpha = [\alpha m, \frac{1}{\alpha}m]$$

Where m is the mean value of $\tilde{M}$ and $\alpha \in [0, 1]$. This fuzzy number may be written as

$M_\alpha = [\underline{\tilde{M}}, \overline{M}]$ where $\underline{M}$ refers to the greatest lower bound of M_α and $\overline{M}$ to the least upper bound of M_α.

Remark (2):

Similar to the second approach given in remark (1), one can fuzzyfy any crisp or nonfuzzy function f, by letting:

$$\underline{f}(x) = \alpha f(x), \quad \bar{f}(x) = \frac{1}{\alpha}f(x) \qquad , x \in X, \alpha \in (0, 1]$$

and hence the fuzzy function $\tilde{f}$ in terms of its α -levels is given by $f_\alpha = [\underline{f}, \bar{f}]$.

III. FINITE DIFFERENCE METHOD FOR SOLVING FUZZY LINEAR BOUNDARY VALUE PROBLEMS:

In this paper, we consider the finite difference method for the linear second order fuzzy BVP:

$$y'' = p(x)y' + q(x)y + r(x), \qquad a \leq x \leq b \qquad \ldots\ldots\ldots\ldots\ldots(1)$$
$$y(a) \simeq \tilde{\alpha}, \quad y(b) \simeq \tilde{\beta}$$

requires that the difference approximations be used for approximating both y′ and y″. To accomplish this, we select an integer N > 0 and divide the interval [a, b] into N + 1 equal subintervals, whose end points are the mesh points $x_i = a + ih$, for i = 0, 1, …, N + 1; where $h = \frac{b-a}{N+1}$. Choosing the constant h in this manner will facilitate the application of a matrix algorithm, which in this form will require of solving a linear system that involves N×N matrix.

In this case, the parametric equations related to eq.(1) are given by:

$$\underline{y}'' = p(x)\underline{y}' + q(x)\underline{y} + r(x), \qquad a \leq x \leq b$$

With boundary conditions:

$$\underline{y}_\alpha(a) = \tilde{\alpha} - \sqrt{1-\alpha}, \quad and \quad \underline{y}_\alpha(b) = \tilde{\beta} - \sqrt{1-\alpha}$$

and

$$\overline{y}'' = p(x)\overline{y}' + q(x)\overline{y} + r(x), \qquad a \leq x \leq b$$

with boundary conditions:

$$\overline{y}_\alpha(a) = \tilde{\alpha} + \sqrt{1-\alpha}, \quad and \quad \overline{y}_\alpha(b) = \tilde{\beta} + \sqrt{1-\alpha}$$

We will carry the discussion for lower case problem and we notify that similar approach is carried in upper case.

$$\underline{y}''(x_i) = p(x_i)\underline{y}'(x_i) + q(x_i)\underline{y}(x_i) + r(x_i) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Expanding y in a third-degree Taylor polynomial about xi evaluated at x_{i+1} and x_{i-1}, and assuming $\underline{y} \in C4[x_{i-1}, x_{i+1}]$, we have:

$$y(x_{i+1}) = \underline{y}(x_i + h)$$
$$= \underline{y}(x_i) + h\underline{y}'(x_i) + \frac{h^2}{2}\underline{y}''(x_i) + \frac{h^3}{6}\underline{y}'''(x_i) + \frac{h^4}{24}\underline{y}''''(\xi_i^+)$$

for some point ξi+, $x_i < \xi i+ < x_{i+1}$, and:

$$\underline{y}(x_{i-1}) = \underline{y}(x_i - h)$$
$$= \underline{y}(x_i) + h\underline{y}'(x_i) + \frac{h^2}{2}\underline{y}''(x_i) + \frac{h^3}{6}\underline{y}'''(x_i) + \frac{h^4}{24}\underline{y}''''(\xi_i^+)$$

for some point ξi−, $x_{i-1} < \xi i- < x_i$.

If these equations are added together, the terms involving $\underline{y}'$(xi) and $\underline{y}'''$(xi) are eliminated, and a simple algebraic manipulation gives:

$$\underline{y}''(x_i) = \frac{1}{h^2}\left[\underline{y}(x_{i+1}) - 2\underline{y}(x_i) + \underline{y}(x_{i-1})\right] - \frac{h^2}{24}\left[\underline{y}^{(4)}(\xi_i^+) + \underline{y}^{(4)}(\xi_i^-)\right]$$

The intermediate value theorem can be used to simplify the last equation even further:

$$\underline{y}''(x_i) = \frac{1}{h^2}\left[\underline{y}(x_{i+1}) - 2\underline{y}(x_i) + \underline{y}(x_{i-1})\right] - \frac{h^2}{12}\underline{y}^{(4)}(\xi_i) \quad \text{...................................(3)}$$

for some point ξi, $x_{i-1} < \xi i < x_{i+1}$.

Equation (3) is called the central difference formula for $\underline{y}''(x_i)$.

A central-difference formula for $\underline{y}'(x_i)$ can be obtained in a similar manner, resulting in:

$$\underline{y}'(x_i) = \frac{1}{2h}\left[\underline{y}(x_{i+1}) - \underline{y}(x_{i-1})\right] - \frac{h^2}{6}\underline{y}'''(\eta i) \quad \text{.......................................(4)}$$

for some ηi, where $x_{i-1} < \eta i < x_{i+1}$.

The use of these central-difference formulas in eq.(2) results in the equation:

$$\frac{1}{h^2}\left[\underline{y}(x_{i+1}) - 2\underline{y}(x_i) + \underline{y}(x_{i-1})\right] =$$
$$p(x_i)\frac{1}{2h}\left[\underline{y}(x_{i+1}) - \underline{y}(x_{i-1})\right] + q(x_i)\underline{y}(x_i) + r(x_i) - \frac{h^2}{12}\left[2\,p(x_i)\,\underline{y}''(\eta i) - \underline{y}^{(4)}(\xi_i)\right]$$

A finite difference method wit truncation error of order o(h2) results by using this equation together with the boundary conditions $\underline{y}(a) = \tilde{\alpha} - \sqrt{1-\alpha}$ and $\underline{y}(b) = \tilde{\beta} - \sqrt{1-\alpha}$, to define:

$$\underline{y}_0(a) = \tilde{\alpha} - \sqrt{1-\alpha}, \quad \text{and} \quad \underline{y}_{N+1}(b) = \tilde{\beta} - \sqrt{1-\alpha}$$

And

$$\frac{1}{h^2}\left[2\,\underline{y}_i - \underline{y}_{i+1} + \underline{y}_{i-1}\right] + \frac{1}{2h}p(x_i)\left[\underline{y}_{i+1} - \underline{y}_{i-1}\right] + q(x_i)\underline{y}_i = -r(x_i)\text{.........(5)}$$

for each i = 1, 2, …, N. In the form we will consider eq.(5) is rewritten as:

$$-\left[1 + \frac{h}{2}p(x_i)\right]\underline{y}_{i-1} + [2 + h^2 q(x_i)]\underline{y}_i - \left[1 - \frac{h}{2}p(x_i)\right]\underline{y}_{i+1} = -h^2 r(x_i)$$

and the resulting system of algebraic equations is expressed in the tridiagonal N×N matrix form:

$$A\underline{y} = b\text{...(6)}$$

where:

$$A = \begin{bmatrix} 2+h^2q(x_1) & -1+\dfrac{h}{2}p(x_1) & 0 & & & 0 \\ -1-\dfrac{h}{2}p(x_2) & 2+h^2q(x_2) & -1+\dfrac{h}{2}p(x_2) & & & \\ 0 & -1-\dfrac{h}{2}p(x_i) & 2+h^2q(x_i) & -1+\dfrac{h}{2}p(xi) & & \\ & & & & & 0 \\ & & & & & -1+\dfrac{h}{2}p(x_{n-1}) \\ 0 & & & 0 & -1-\dfrac{h}{2}p(x_n) & 2+h^2q(x_n) \end{bmatrix}$$

$$\underline{y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \underline{y}_i \\ \vdots \\ \underline{y}_{N-1} \\ \underline{y}_N \end{bmatrix}, \quad b = \begin{bmatrix} -h^2r(x_1)+\left(1+\dfrac{h}{2}p(x_1)\right)\underline{y}_0 \\ -h^2r(x_2) \\ \vdots \\ -h^2r(x_i) \\ \vdots \\ -h^2r(x_{N-1}) \\ -h^2r(x_N)+\left(1-\dfrac{h}{2}p(x_N)\right)\underline{y}_{N+1} \end{bmatrix}$$

IV. Illustrative Example

Consider the fuzzy BVP:

$$y'' = \frac{-2}{x}y' + \frac{2}{x^2}y + x^2, \qquad 1 \le x \le 2 \qquad \text{.......................................(7)}$$

$$y(1) \simeq 1, \quad y(2) \simeq 2 \qquad \text{..(8)}$$

For this example, we will use N = 9, so that h = 0.1. In this case, the parametric equations related to eqs.(7) and (8), are:

$$\underline{y}''_{\alpha}(x) = \frac{-2}{x}\underline{y}'_{\alpha}(x) + \frac{2}{x^2}\underline{y}_{\alpha}(x) + x^2 \qquad , x \in [1,2]$$

With boundary conditions:

$$\underline{y}_{\alpha}(1) = 1 - \sqrt{1-\alpha}, \quad \text{and} \quad \underline{y}_{\alpha}(2) = 2 - \sqrt{1-\alpha}$$

which is a non fuzzy BVP.

By using eq.(4), we get:

$$\frac{\underline{y}(x_{i+1})-2\underline{y}(x_i)+\underline{y}(x_{i-1})}{h^2} = \frac{-2}{x_i}\left[\frac{\underline{y}(x_{i+1})-\underline{y}(x_{i-1})}{2h}\right] + \frac{2}{x_i^2}\underline{y}(x_i) + x_i^2$$

Now, define:

$$\underline{y}_0(1) = 1 - \sqrt{1-\alpha}, \quad \text{and} \quad \underline{y}_{10}(2) = 2 - \sqrt{1-\alpha}$$

Hence:

$$z\frac{y_i - y_{i+1} + y_{i-1}}{h^2} - \frac{2}{x_i} + \frac{2}{x_i^2} \underline{y}_i = -x_i^2 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(9)$$

For each i = 1, 2, …, 9.

In this form, we will consider eq.(9) which can be written as:

$$-\left[1 - \frac{h}{x_i}\right]\underline{y}_{i-1} + \left[2 + \frac{2}{x^2_i}h^2\right]\underline{y}_i - \left[1 + \frac{h}{x_i}\right]\underline{y}_{i+1} = -x_i^2 h2$$

and the resulting system of algebraic equations may be expressed in the triadiagonal 9×9 matrix form:

$$A\underline{y} = b \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(10)$$

Where:

$$A = \begin{bmatrix}
2+\frac{2}{x_1^2}h^2 & -1-\frac{h}{x_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1+\frac{h}{x_2} & 2+\frac{2}{x_2^2}h^2 & -1-\frac{h}{x_2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1+\frac{h}{x_3} & 2+\frac{2}{x_3^2}h^2 & -1-\frac{h}{x_3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1+\frac{h}{x_4} & 2+\frac{2}{x_4^2}h^2 & -1-\frac{h}{x_4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1+\frac{h}{x_5} & 2+\frac{2}{x_5^2}h^2 & -1-\frac{h}{x_5} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1+\frac{h}{x_6} & 2+\frac{2}{x_6^2}h^2 & -1-\frac{h}{x_6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1+\frac{h}{x_7} & 2+\frac{2}{x_7^2}h^2 & -1-\frac{h}{x_7} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1+\frac{h}{x_8} & 2+\frac{2}{x_8^2}h^2 & -1-\frac{h}{x_8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+\frac{h}{x_9} & 2+\frac{2}{x_9^2}h^2
\end{bmatrix}$$

$$\underline{y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{y}_3 \\ \underline{y}_4 \\ \underline{y}_5 \\ \underline{y}_6 \\ \underline{y}_7 \\ \underline{y}_8 \\ \underline{y}_9 \end{bmatrix}, \quad b = \begin{bmatrix} -h^2 x_1^2 + \left(1 - \frac{h}{x_1}\right)\left(1 - \sqrt{1-\alpha}\right) \\ -h^2 x_2^2 \\ -h^2 x_3^2 \\ -h^2 x_4^2 \\ -h^2 x_5^2 \\ -h^2 x_6^2 \\ -h^2 x_7^2 \\ -h^2 x_8^2 \\ -h^2 x_9^2 + \left(1 + \frac{h}{x_9}\right)\left(2 - \sqrt{1-\alpha}\right) \end{bmatrix}$$

We can carry similar calculations as it is followed for lower case of solution and find the upper case of solution from the following system of algebraic equations:

$$A\bar{y} = b \dots\dots\dots(11)$$

Where

$$A = \begin{bmatrix} 2+\dfrac{2}{x_1^2}h^2 & -1-\dfrac{h}{x_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1+\dfrac{h}{x_2} & 2+\dfrac{2}{x_2^2}h^2 & -1-\dfrac{h}{x_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1+\dfrac{h}{x_3} & 2+\dfrac{2}{x_3^2}h^2 & -1-\dfrac{h}{x_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1+\dfrac{h}{x_4} & 2+\dfrac{2}{x_4^2}h^2 & -1-\dfrac{h}{x_4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1+\dfrac{h}{x_5} & 2+\dfrac{2}{x_5^2}h^2 & -1-\dfrac{h}{x_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1+\dfrac{h}{x_6} & 2+\dfrac{2}{x_6^2}h^2 & -1-\dfrac{h}{x_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1+\dfrac{h}{x_7} & 2+\dfrac{2}{x_7^2}h^2 & -1-\dfrac{h}{x_7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1+\dfrac{h}{x_8} & 2+\dfrac{2}{x_8^2}h^2 & -1-\dfrac{h}{x_8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+\dfrac{h}{x_9} & 2+\dfrac{2}{x_9^2}h^2 \end{bmatrix}$$

$$\underline{y} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \\ \bar{y}_4 \\ \bar{y}_5 \\ \bar{y}_6 \\ \bar{y}_7 \\ \bar{y}_8 \\ \bar{y}_9 \end{bmatrix}, \quad b = \begin{bmatrix} -h^2x_1^2 + \left(1-\dfrac{h}{x_1}\right)\left(1+\sqrt{1-\alpha}\right) \\ -h^2x_2^2 \\ -h^2x_3^2 \\ -h^2x_4^2 \\ -h^2x_5^2 \\ -h^2x_6^2 \\ -h^2x_7^2 \\ -h^2x_8^2 \\ -h^2x_9^2 + \left(1+\dfrac{h}{x_9}\right)\left(2+\sqrt{1-\alpha}\right) \end{bmatrix}$$

Where α is a parameter between 0 and 1, and when $\alpha = 1$, the results are given in table (1) which represented the crisp solution .

TABLE 1

The crisp solution .

xi	$\underline{y}(x_i)$	$\bar{y}(x_i)$
1	1	1
1.1	1.012	1.012
1.2	1.127	1.127
1.3	1.244	1.244
1.4	1.3747	1.3747
1.5	1.455	1.455
1.6	1.561	1.561
1.7	1.673	1.673
1.8	1.781	1.781
1.9	1.895	1.895
2	2	2

Hence $y\alpha = [\underline{y}\alpha, \bar{y}\alpha]$ is the solution of the original fuzzy BVP.

V. CONCLUSIONS:

From the present study of this paper, we may conclude the following:

1. The accuracy of the results may be checked with $\alpha = 1$, in which the upper and lower solutions must be equal.

2. The crisp solution or the solution of the nonfuzzy boundary value problem is obtained from the fuzzy solution by setting $\alpha = 1$, and therefore fuzzy boundary value problems may be considered as a generalization to the nonfuzzy boundary value problems.

3. The finite difference method proved its validity and accurate results in solving fuzzy boundary value problems.

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