Generalized Identities on Derivatives of $p(y)$-Fibonacci Polynomial And $p(y)$-Lucas Polynomial

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Abstract — *The Fibonacci and Lucas Polynomials are well known for having interesting and amazing properties and identities. In this paper we introduce* $\mathbf{p(y)}$*-Fibonacci and* $\mathbf{p(y)}$*-Lucas polynomials where* $\mathbf{p(y)}$ *is a polynomial with real coefficients and some basic identities are derived by using generating function of these polynomials.*

Keywords — $\mathbf{p(y)}$*-Fibonacci polynomial,* $\mathbf{p(y)}$*-Lucas polynomial.*

I. INTRODUCTION

Fibonacci and Lucas polynomials are firmly related and generally explored. The Fibonacci and Lucas polynomials are outstanding for having intriguing and stunning properties. These polynomials are of incredible significance in the investigation of numerous subjects, for example Approximation theory, Combinatorics, Algebra, Geometry, Statics and Number theory itself. Some of the captivating properties of these polynomials have been examined in [1], [2], [4]. Omprakash Sikhwal [6] in his paper obtained some identities of generalized Fibonacci Polynomials by the method of generating functions, in this paper we have obtained some identities involving derivatives of $p(y)$-Fibonacci and $p(y)$-Lucas polynomials by method of generating function.

II. $p(y)$-FIBONACCI POLYNOMIAL

$p(y)$-Fibonacci polynomials defined by the recurrence relation as

$$F_{p,n+1}(y) = p(y)F_{p,n}(y) + F_{p,n-1}(y), \text{n} \geq 1 \tag{2.1}$$

with initial condition $F_{p,0}(y) = 0, F_{p,1}(y) = 1$

where, $p(y)$ is a polynomial with real coefficients

first few $p(y)$-Fibonacci polynomials are

$$F_{p,1}(y) = 1$$
$$F_{p,2}(y) = p(y)$$
$$F_{p,3}(y) = p^2(y) + 1$$
$$F_{p,4}(y) = p^3(y) + 2p(y)$$
$$F_{p,5}(y) = p^4(y) + 3p^2(y) + 1 \text{ and so, on}$$

For $p(y) = x$ in equation (2.1) we obtain the Catalan's Fibonacci Polynomials as

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x), \text{n} \geq 1 \tag{2.2}$$

with initial condition $F_0(x) = 0, F_1(x) = 1$

$$F_1(x) = 1$$
$$F_2(x) = x$$
$$F_3(x) = x^2 + 1$$
$$F_4(x) = x^3 + 2x$$
$$F_5(x) = x^4 + 3x^2 + 1 \text{ and so, on}$$

For $p(y) = 2x$ in equation (2.1) we obtain Byrd's Fibonacci polynomials as

$$F_n(x) = 2xF_{n-1}(x) + F_{n-2}(x), \text{n} \geq 2 \tag{2.3}$$

with initial condition $F_0(x) = 0, F_1(x) = 1$

The generating function $g_F(s)$ of the $p(y)$-Fibonacci polynomials is defined by

$$g_F(s) = \sum_{n=0}^{\infty} F_{p,n}(y)s^n = s[1 - p(y)s - s^2]^{-1} \tag{2.4}$$

where, $g_F(s)$ is a formal power series.
On differentiating equation (2.1) with respect to y we get

$$F'_{p,n+1}(y) = p(y)\,F'_{p,n}(y) + F_{p,n}(y)p'(y) + F'_{p,n-1}(y), n \geq 1 \tag{2.5}$$

Theorem 2.1. If $F_{p,n}(y)$ is the $p(y)$-Fibonacci polynomial, then prove that

$$(n-1)p'(y)F_{p,n}(y) = p(y)F'_{p,n}(y) + 2F'_{p,n-1}(y),\ n \geq 1$$

Proof: By generating function of $p(y)$-Fibonacci polynomial we have

$$\sum_{n=0}^{\infty} F_{p,n}(y)s^n = s[(1 - (p(y) + s)s]^{-1} \tag{2.6}$$

Differentiating partially both sides with respect to s, we get

$$\sum_{n=0}^{\infty} nF_{p,n}(y)s^{n-1} = s[(1 - (p(y) + s)s]^{-2}[2s + p(y)] + [(1 - (p(y) + s)s]^{-1} \tag{2.7}$$

again, differentiating equation (2.5) partially with the respect to y, we get

$$\sum_{n=0}^{\infty} F'_{p,n}(y)s^n = s[(1 - (p(y) + s)s)]^{-2}[sp'(y)]$$

$$\sum_{n=0}^{\infty} F'_{p,n}(y)s^{n-1} = s[(1 - (p(y) + s)s)]^{-2}(p'(y)$$

$$\frac{1}{p'(y)}\sum_{n=0}^{\infty} F'_{p,n}(y)s^{n-1} = s[(1 - (p(y) + s)s)]^{-2} \tag{2.8}$$

by equation (2.7) and (2.8) we get

$$\sum_{n=0}^{\infty} nF_{p,n}(y)s^{n-1} = \frac{(2s + p(y))}{p'(y)}\sum_{n=0}^{\infty} F'_{p,n}(y)s^{n-1} + [1 - (p(y) + s)s]^{-1}$$

by using equation $(1.2.6)$, we get

$$\sum_{n=0}^{\infty} nF_{p,n}(y)s^{n-1} = \frac{(2s + p(y))}{p'(y)}\sum_{n=0}^{\infty} F'_{p,n}(y)s^{n-1} + \sum_{n=0}^{\infty} F_{p,n}(y)s^{n-1}$$

$$\sum_{n=0}^{\infty} np'(y)F_{p,n}(y)\,s^{n-1} = 2\sum_{n=0}^{\infty} F'_{p,n}(y)\,s^n + p(y)\sum_{n=0}^{\infty} F'_{p,n}(y)\,s^{n-1} + p'(y)\sum_{n=0}^{\infty} F_{p,n}(y)\,s^{n-1} \tag{2.9}$$

on comparing the coefficient of s^{n-1} from equation (2.9), we get

$$np'(y)F_{p,n}(y) = 2F'_{p,n-1}(y) + p(y)F'_{p,n}(y) + p'(y)F_{p,n}(y)$$

$$(n-1)p'(y)F_{h,n}(y) = 2F'_{p,n-1}(y) + p(y)F'_{p,n}(y) \tag{2.10}$$

Theorem 2.2 If $F_{p,n}(y)$ is the $p(y)$-Fibonacci polynomial, then prove that

(i) $np'(y)F_{p,n}(y) = F'_{p,n+1}(y) + F'_{p,n-1}(y)$

(ii) $p(y)F'_{p,n}(y) = 2F'_{p,n+1}(y) - (n+1)p'(y)\,F'_{p,n}(y)$

Proof of part (i): By equation (2.4) we have

$$F'_{h,n+1}(y) - F'_{p,n-1}(y) = p'(y)F_{p,n}(y) + p(y)F'_{p,n}(y)$$

$$F'_{p,n+1}(y) + F'_{p,n-1}(y) = 2F'_{p,n-1}(y) + p(y)\,F'_{p,n}(y) + p'(y)F_{p,n}(y) \tag{2.11}$$

by using equations (2.10) and (2.11), we get

$$F'_{p,n+1}(y) + F'_{h,n-1}(y) = (n-1)p'(y)\,F_{p,n}(y) + p'(y)F_{p,n}(y)$$

$$F'_{p,n+1}(y) + F'_{p,n-1}(y) = (n - 1 + 1)p'(y)F_{p,n}(y)$$

$$F'_{p,n+1}(y) + F'_{p,n-1}(y) = np'(y)\,F_{p,n}(y)$$

$$np'(y)F_{p,n}(y) = F'_{p,n+1}(y) + F'_{p,n-1}(y) \tag{2.12}$$

Proof of part (ii): By equation (2.10) we have

$(n-1)p'(y)F_{p,n}(y) = 2F'_{p,n-1}(x) + p(y)F'_{p,n}(y)$
$(n-1)p'(y)F_{p,n}(y) - p(y)F'_{p,n}(y) = 2F'_{p,n-1}(y)$
$\frac{1}{2}\left[(n-1)p'(y)F_{p,n}(y) - p(y)F'_{p,n}(y)\right] = F'_{p,n-1}(y)$ (2.13)

by using equations (2.12) and (2.13), we get

$np'(y)F_{p,n}(y) = F'_{p,n+1}(y) + \frac{1}{2}\left[(n-1)p'(y)F_{p,n}(y) - p(y)F'_{p,n}(y)\right]$
$2np'(y)F_{p,n}(y) = 2F'_{p,n+1}(y) + \left[(n-1)p'(y)F_{p,n}(y) - p(y)F'_{p,n}(y)\right]$
$(n+1)p'(y)F_{p,n}(y) = 2F'_{p,n+1}(y) - p(y)F'_{p,n}(y)$
$p(y)F'_{p,n}(y) = 2F'_{p,n+1}(y) - (n+1)p'(y)F_{p,n}(y)$ (2.14)

Theorem 2.3 If $F_{p,n}(y)$ is the $p(y)$-Fibonacci polynomial, then prove that

$(n-1)F'_{p,n+1}(y) - (n+1)F'_{p,n-1}(y) = np(y)F'_{p,n}(y)$

Proof: By using equation (2.5), we have

$F'_{p,n+1}(y) = p(y)\, F'_{p,n-1}(y) + F_{p,n-1}(y)p'(y) + F'_{p,n-1}(y)$
$F'_{p,n+1}(y) - p(y)\, F'_{p,n}(y) - F'_{p,n-1}(y) = F_{p,n-1}(y)p'(y)$

by multiplying n on both sides, we get

$n\left[F'_{p,n+1}(y) - p(y)\, F'_{p,n}(y) - F'_{p,n-1}(y)\right] = nF_{p,n-1}(y)p'(y)$ (2.15)

by using equations (2.12) and (2.15), we get

$n\left[F'_{p,n+1}(y) - p(y)\, F'_{p,n}(y) - F'_{p,n-1}(y)\right] = F'_{p,n+1}(y) + F'_{p,n-1}(y)$
$(n-1)F'_{p,n+1}(y) - (n+1)\, F'_{p,n-1}(p) = np(y)F'_{p,n}(y)$ (2.16)

III. $p(y)$-LUCAS POLYNOMIAL

The $p(y)$ - Lucas polynomials are defined as

$L_{p,n+1}(y) = p(y)L_{p,n}(y) + L_{p,n-1}(y),\ \ n \geq 1$ (3.1)

with initial conditions $L_{p,0}(y) = 2,\ L_{p,1}(y) = p(y)$, where $p(y)$ is a polynomial in variable y having real coefficients

By taking $p(y) = x$ in equation (3.1) we get Lucas polynomials

$L_{n+1}(x) = xL_n(x) + L_{n-1}(x),\ \ n \geq 1$

with initial conditions $L_{p,0}(x) = 2,\ L_{p,1}(x) = x$

the generating function of the $p(y)$-Lucas Polynomials is defined as

$$g_L(s) = \sum_{n=0}^{\infty} L_{p,n}(y)s^n = (2 - p(y)s)[1 - (p(y) + s)s]^{-1} \quad (3.2)$$

on differentiating equation (3.1) we obtained

$L'_{p,n+1}(y) = p(x)\, L'_{p,n}(y) + p'(x)\, L_{p,n}(y) + L'_{p,n-1}(y)$ (3.3)

Theorem 3.1 If $F_{p,n}(y)$ is the $p(y)$-Fibonacci polynomial and $L_{p,n}(y)$ is the $p(y)$-Lucas polynomial, then prove that

$np'(y)L_{p,n}(y) - p(y)\, L'_{p,n}(y) = 2[L'_{p,n-1}(y) + p'(y)F_{p,n-1}(y)]$

Proof: We know that generating function of $p(y)$-Lucas polynomial given by

$$\sum_{n=0}^{\infty} L_{p,n}(y)s^n = (2 - p(y)s)[1 - (p(y) + s)s]^{-1} \quad (3.4)$$

On differentiating equation (3.4) partially with respect to s, we get

$$\sum_{n=0}^{\infty} nL_{p,n}(y)s^{n-1} = (2 - p(y)s)[1 - (p(y) + s)s]^{-2}(p(y) + 2s) - p(y)[1 - (p(y) + s)s]^{-1} \quad (3.5)$$

again, differentiating equation (3.4) partially with respect to y

$$\sum_{n=0}^{\infty} L'_{p,n}(y)s^n = (2 - p(y)s)[1 - (p(y) + s)s]^{-2}(p'(y)s) + [1 - (p(y) + s)s]^{-1}(-p'(y)s)$$

$$\sum_{n=0}^{\infty} L'_{p,n}(y)s^n = sp'(y)[(2 - p(y)s)[1 - (p(y) + s)s]^{-2} - [1 - (p(y) + s)s]^{-1}]$$

$$\frac{1}{p'(y)}\sum_{n=0}^{\infty} L'_{p,n}(y)s^{n-1} = (2-p(y)s)[1-(p(y)+s)s]^{-2} - [1-(p(y)+s)s]^{-1}$$

$$\frac{1}{p'(y)}\sum_{n=0}^{\infty} L'_{p,n}(y)s^{n-1} - [1-(p(y)+s)s]^{-1} = (2-p(y)s)[[1-(p(y)+s)s]^{-2} \tag{3.6}$$

by using equations (3.5) and (3.6), we get

$$\sum_{n=0}^{\infty} nL_{p,n}(y)s^{n-1}$$
$$= (p(y)+2s)\left[\frac{1}{p'(y)}\sum_{n=0}^{\infty} L'_{p,n}(y)s^{n-1} - [1-(p(y)+s)s]^{-1}\right]$$
$$- p(y)[1-(p(y)+s)s]^{-1}$$

$$\sum_{n=0}^{\infty} nL_{p,n}(y)s^{n-1}$$
$$= \frac{p(y)}{p'(y)}\sum_{n=0}^{\infty} L'_{p,n}(y)s^{n-1} + \frac{2}{p'(y)}\sum_{n=0}^{\infty} L'_{h,n}(y)s^{n}$$
$$+ [(p(y)+2s)-p(y)][1-(p(y)+s)s]^{-1}$$

$$\sum_{n=0}^{\infty} nL_{p,n}(y)s^{n-1} = \frac{p(y)}{p'(y)}\sum_{n=0}^{\infty} L'_{p,n}(y)s^{n-1} + \frac{2}{p'(y)}\sum_{n=0}^{\infty} L'_{p,n}(y)s^{n} + 2s[1-(p(y)+s)s]^{-1}$$

by using equation (2.4)

$$p'(y)\sum_{n=0}^{\infty} nL_{p,n}(y)s^{n-1} = p(y)\sum_{n=0}^{\infty} L'_{p,n}(y)s^{n-1} + 2\sum_{n=0}^{\infty} L'_{p,n}(y)s^{n} + 2p'(y)\sum_{n=0}^{\infty} F_{p,n}(y)s^{n}$$

by equating coefficients of s^{n-1} on both sides, we obtained

$$np'(y)L_{p,n}(y) = p(y)L'_{p,n}(y) + 2L'_{p,n-1}(y) + 2p'(y)F_{p,n-1}(y)$$

$$np'(y)L_{p,n}(y) - p(y)L'_{p,n}(y) = 2[L'_{p,n-1}(y) + p'(y)F_{p,n-1}(y)] \tag{3.7}$$

Theorem 3.2 If $F_{p,n}(y)$ is the $p(y)$-Fibonacci polynomial and $L_{p,n}(y)$ is the $p(y)$-Lucas polynomial, then prove that

$$L'_{p,n+1}(y) + L'_{p,n-1}(y) = (n-1)p'(y)L_{p,n}(y) - 2p'(y)F_{p,n-1}(y)$$

Proof: By equation (3.3), we have

$$L'_{p,n+1}(y) - L'_{p,n-1}(y) = p(y)L'_{p,n}(y) + p'(y)\,L_{p,n}(y)$$

$$L'_{p,n+1}(y) + L'_{p,n-1}(y) - L'_{p,n-1}(y) - L'_{p,n-1}(y) = p(y)L'_{p,n}(y) + p'(y)\,L_{p,n}(y)$$

$$L'_{p,n+1}(y) + L'_{p,n-1}(y) = 2L'_{p,n-1}(y) + p(y)L'_{p,n}(y) + p'(y)\,L_{p,n}(y)$$

by using equation (3.7) we get

$$L'_{p,n+1}(y) + L'_{p,n-1}(y) = np'(y)L_{p,n}(y) - 2p'(y)F_{p,n-1}(y) + p'(y)\,L_{p,n}(y)$$

$$L'_{p,n+1}(y) + L'_{p,n-1}(y) = (n+1)p'(y)L_{p,n}(y) - 2p'(y)F_{p,n-1}(y)$$

CONCLUSION

In this paper we have establish and prove some identities involving $p(y)$-Fibonacci and $p(y)$-Lucas Polynomials are described with derivation by using generating function of these polynomials.

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