About Enclave Inclusive Sets In Graphs

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Abstract *- We introduce the concept of enclave inclusive set in graphs in this paper. A set of vertices of a graph is called an enclave inclusive set if contains an enclave point. We prove that a set of vertices is a minimal enclave set if and only if its compliment is a maximal non dominating set. We observe that the close neighbourhood of a vertex with minimum degree is a minimum enclave inclusive set. We also prove that if* v*is a vertex of a graph. Then enclave inclusive number of* $G - v$*is less than the enclave inclusive number of* G*if and only if there is a neighbour* u *of* v*such that* $d(u)$ *is minimum. We deduce that for a graph* G *without isolated vertices there are at least* $\delta(G)$*vertices such that removal of any one of them reduces the enclave inclusive number of the graph. We further prove that if* $e = \{uv\}$ *is an edge of the graph* G*. Then enclave inclusive number of* $G - e$ *is less than the enclave inclusive number of* G *if and only if* $d(u)$ *is minimum or* $d(v)$ *is minimum. Finally, we observe that if* G *is a* k*-regular graph* $(k \geq 1)$*. Then removal of any vertex or any edge reduce the enclave inclusive number of the graph.*

Keywords *- enclave point, enclave inclusive set, minimum enclave inclusive set, minimal enclave inclusive set, upper enclave inclusive number.*

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I. INTRODUCTION

If S is a set of vertices of $G$and $v \in S$. Then we said to be an enclave point of Sif $N[v] \subset S$. A set S of vertices of G is said to be an enclave inclusive set if S contains an enclave vertex. To be an enclave inclusive set is a super hereditary property but it is not hereditary property. An enclave inclusive set with minimum cardinality is called a minimal enclave inclusive set and its cardinality is called enclave inclusive number of the graph. We prove that the close neighbour of a vertex with minimum degree is a minimum enclave inclusive set and the enclave inclusive number of any graph is $\delta(G)$.

We consider two operation on a graph and observe there effect on enclave inclusive number of the graph. Infect, we prove that the removal of a vertex reduces the enclave inclusive number if it has a neighbour with minimum degree. Similarly, we prove that the removal on edge reduces the enclave inclusive number if one of its end vertices has minimum degree.

II. PRELIMINARIES AND NOTATIONS

If G is a graph then $V(G)$ denotes the vertex set of the graph G and $E(G)$ denotes the edge set of the graph G. If v is vertex of the graph G then $G - v$ is the subgraph of G induced by all the vertices different from v.
We will consider only simple undirected graphs with finite vertex set.

III. DEFINITIONS AND EXAMPLES

Definition 3.1 **:(Non dominating set) :**
A subset Sof V(G) , which is not dominating set then S is called a non dominating set.

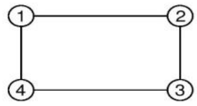
If S is a non dominating set and T ⊂ S. Then T is a non dominating set. Therefore to be non dominating set is a hereditary property.

Definition 3.2 **:(maximalNon dominating set) :**
A non-dominating is said to be maximal non dominating set if it is not properly contain any other non-dominating set.

Equivalently, a non-dominating set S is a maximal non dominating set if for each $v \in V(G) - S$, $S \cup \{v\}$ is a dominating set.

A non-dominating set S is said to be a maximal non dominating set if $S \cup \{v\}$ is a dominating set for every $v \in V(G) - S$.

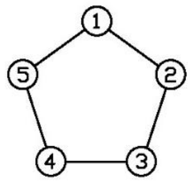
Example 1: Consider the cycle graph C_4 with 4 vertices $\{1, 2, 3, 4\}$



All single sets are non dominating set $\{1\}, \{2\}, \{3\}, \{4\}$ and $\{1, 2\}, \{1, 3\}, \{1, 4\}$ are dominating sets.

Therefore $v_1 = \{1\}$ is a maximal non dominating set.

Example 2: Consider the cycle graph C_5 with 5 vertices $\{1, 2, 3, 4, 5\}$



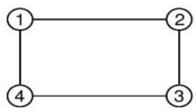
$\{1, 2\}$ is a maximal non dominating set. Cardinality of maximum non dominating set $\{1, 2\} = 2$.

$\{1, 5\}$is a maximal non dominating set and cardinality of maximum non dominating set $\{1, 2, 3\} = 3$.

***Definition 3.3* :(enclave point) :**
Let G be a graph, $S \subset V(G)$and $v \in S$. Then v is an enclave point of S if $N[v] \subset S$.

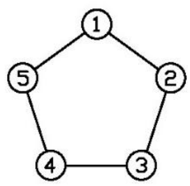
***Definition 3.4* :(enclave inclusive set) :**
A set S is said to be an enclave inclusive set if S contain an enclave point.

Example 3: Consider the cycle graph C_4 with 4 vertices $\{1, 2, 3, 4\}$



Let $S = \{1, 2, 3\}$
$N[2] = \{1, 2, 3\}$then 2 is enclave point of S.
Therefore $S = \{1, 2, 3\}$ is an enclave inclusive set.

Example 4: Consider the cycle graph C_5 with 5 vertices $\{1, 2, 3, 4, 5\}$



Let $S = \{1, 2, 3\}$ is an enclave inclusive set of S and 2 is an enclave point of S.

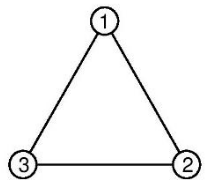
***Definition 3.5* :(minimal enclave inclusive set) :**

An enclave inclusive set S is said to be a minimal enclave inclusive set if no proper subset of S is an enclave inclusive set.

***Definition 3.6* :(minimum enclave inclusive set) :**

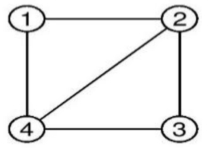
Let G be a graph and enclave inclusive set with minimum cardinality is called minimum enclave inclusive set and its cardinality is called enclave inclusive number of the graph and it is denoted as $e_i(G)$.

Example 5: Consider the cycle graph C_3 with 3 vertices $\{1 , 2 , 3\}$



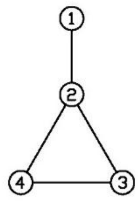
Let $S = \{1 , 2 , 3\}$ then S is a minimum enclave inclusive set and $e_i(G) = 3$.

Example 6: Consider the graph G with 4 vertices $\{1 , 2 , 3 , 4\}$



Let $S = \{1 , 2 , 4\}$ then S is a minimum enclave inclusive set and $e_i(G) = 3$.

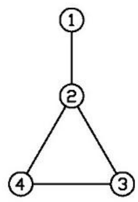
Example 7: Consider the graph$G$ with 4 vertices $\{1 , 2 , 3 , 4\}$



Let $S = \{1 , 2\}$ then S is a minimum enclave inclusive set and $e_i(G) = 2$.
Let $T = \{2, 3, 4\}$. It is obvious that T is a enclave inclusive set.
Note that T is not a minimum enclave inclusive set. Because $|T| > |S|$.

***Definition 3.7* :(upper enclave inclusive number) :**
Let G be a graph. A minimal enclave inclusive set with maximum cardinality is called $E_i - set$of G. The cardinality of such a set is called the upper enclave inclusive number of the graph and it is denoted as $E_i(G)$.

Example 8: Consider the following graph.



In this graph the set $S = \{2, 3, 4\}$ is a $E_i - set$of G.

Here $E_i(G) = 3$.
Note that the set $T = \{1, 2\}$ is a minimum enclave inclusive set and $e_i(G) = 2$.

Remark: For any graph G , $e_i(G) \leq E_i(G)$.

IV. MAIN RESULT

Proposition 4.1: Let G be a graph and $S \subset V(G)$. Then $V(G) - S$ is an enclave inclusive set if and only if S is a non dominating set.

Proof: Suppose S is an enclave inclusive set of G.
Let v be an enclave point of S. Then v is not adjacent to every vertex of $V(G) - S$because $N(v) \subset S$.
Thus $V(G) - S$ is not a dominating set of G.

Conversely, suppose $V(G) - S$ is not a dominating set.
Therefore there is a vertex $v$in S. Which is not adjacent to any vertex of $V(G) - S$.
Therefore $N(v) \subset S$.
Hence v is an enclave point of S.
Thus S is an enclave inclusive set of G. ▮

First we give the characterization of minimal enclave inclusive set.
Theorem 4.2: Let G be a graph and $S \subset V(G)$. Then S is a minimal enclave inclusive set if for each $v \in S$ the following condition is satisfied.
C: If $x \in S, x \neq v$ then x has a neighbour in $V(G) - S$ or x is adjacent to v.

Proof: First suppose that S is a minimal enclave inclusive set of G.
Let $v \in S$ then $S - \{v\}$ is not an enclave inclusive set.
Therefore if $x \in S - \{v\}$ then x is not an enclave point of $S - \{v\}$.
Therefore x has a neighbour in $V(G) - S$ or x is adjacent to v.

Conversely, suppose the condition is satisfied.
Let $v \in S$. Let $x \in S - \{v\}$.
If x is a neighbour in $V(G) - S$ then it follows that x is not an enclave point of $S - \{v\}$.
If x is adjacent to v then also it follows that x is not an enclave point of $S - \{v\}$.
Thus $S - \{v\}$ does not have any enclave point .
Therefore S is a minimal enclave inclusive set of G. ▮

Proposition 4.3: Let G be a graph and $S \subset V(G)$. Then S is a minimal enclave inclusive set if and only if $V(G) - S$ is a maximal non dominating set.

Proof: First suppose that S be a minimal enclave inclusive set then $V(G) - S$ is a non dominating set.
Let $\in V(G) - (V(G) - S)$. Then $v \in S$
Let $T = V(G) - S$ and let consider the $T \cup \{v\}$.
Let $x \in V(G) - (T \cup \{v\})$then $x \in S - \{v\}$. Since S is a minimal enclave inclusive set.
$x$is adjacent to v or x has a neighbour in T.
Therefore $T \cup \{v\}$ is a dominating set.
Therefore T is a maximal non dominating set.

Conversely, suppose $V(G) - S$ is a maximal non dominating set of G.
Let $v \in S$. Consider the set $T = V(G) - (S \cup \{v\})$.
Let $x \in S$ such that $x \neq v$.
Now T is a dominating set of Gand $x \notin T$.
Therefore x is adjacent to v or x has a neighbour in $V(G) - S$.
By proposition 4.2, S is a minimal enclave inclusive set. ▮

Proposition 4.4: Let G be a graph and $v \in V(G)$ such that $d(v) = \delta(G)$ then $N[v]$ is a minimal enclave inclusive set.

Proof: It is obvious that $N[v]$ is an enclave inclusive set of G.
Let S be any enclave inclusive set of G.

Let x be an enclave point of Sthen $N[x] \subset$ S.
Therefore,$|S| \geq |N[x]| \geq |N[v]|$.
Therefore,$N[v]$ is a minimum enclave inclusive set of G. ▌

Corollary 4.5: For any graph , $e_i(G) = \delta(G) + 1$.

Proof: It is obvious.

Remark: Let G be a graph. It is obvious that V(G) is a enclave inclusive set with maximum cardinality. Therefore, it is not interesting to define an enclave inclusive set with maximum cardinality.

Now we state and prove a necessary and sufficient condition under which the enclave inclusive number decreases when a vertex is removing from the graph.

Proposition 4.6: Let G be a graph and $v \in$ V(G) then $e_i(G - v) < e_i(G)$ if and only if there is a neighbour u of v such that $d(u)$in G $= \delta(G)$.

Proof: suppose $e_i(G - v) < e_i(G)$.
Therefore,$\delta(G - v) + 1 < \delta(\mathrm{G}) + 1$.
Therefore,$\delta(G - v) < \delta(\mathrm{G})$.
Let w be a vertex of $G - v$ such that $d(w)$in $G - v = \delta(G - v) < \delta(\mathrm{G})$.
Therefore $d(w)$in $G \leq \delta(\mathrm{G})$.................... (1)
However, $d(w)$in $G \geq \delta(\mathrm{G})$.................... (2)
From (1) and (2) it follows that
$d(w)$in $G = \delta(\mathrm{G})$.
If w is not a neighbour of v then it would implies that $d(w)$ in $G - v = \delta(\mathrm{G} - \mathrm{v}) = \mathrm{d(w)}$ in $G = \delta(\mathrm{G})$ and this would implies that $\delta(G - v) = \delta(\mathrm{G})$.
Which is a contradiction.
Therefore w is a neighbour of v.
Conversely, suppose there is a neighbour u of v such that $d(u)$in $G = \delta(\mathrm{G})$.
Therefore $d(u)$in $G - v = \delta(G) - 1$.
Claim :$e_i(G - v) < e_i(G)$
Proof of the claim :$e_i(G - v) < \delta(G) - 1$
Then $\delta(G - v) + 1 < \delta(G) - 1$.
Therefore $\delta(G - v) < \delta(G) - 2$.
Let x be a vertex of $G - v$ such that $d(x)$ in $G - v = \delta(G - v)$
Therefore $d(x)$in $G - v < \delta(G) - 2$.
Therefore $d(x)$in $\leq \delta(G) - 1$.
Therefore $d(x)$in $< \delta(G)$.
Which is a contradiction.
Therefore $e_i(G - v) = \delta(G) - 1 < \delta(G) + 1 = e_i(G)$
Therefore $e_i(G - v) < e_i(G)$ ▌

Corollary 4.7: Let G be a graph and $v \in$ V(G) then $e_i(G - v) > e_i(G)$ if and only if for every neighbour u of v in G such that $d(u)$in G $> \delta(G)$.

Proof: The proof follows from above theorem-6.

Remark: From the above theorem-6 and its corollary-7, follows that if G is a graph and $v \in$ V(G).
Then exactly one of two possibilities holds.
(1) $e_i(G - v) > e_i(G)$
(2) $e_i(G - v) > e_i(G)$

Corollary 4.8: Let G be a graph without isolated vertices then there are at least $\delta(G)$ vertices such that removal of each of them decreases enclave inclusive number of the graph.

Proof: Let v be a vertex of G such that $d(v) = \delta(G)$then $d(v) \geq 1$.
If u is any neighbour of $v$then $e_i(G - v) < e_i(G)$.
Thus the removal of any neighbour of v decreases the enclave inclusive number of the graph.
Thus, there are at least $\delta(G)$ vertices such that removal of each of them decreases enclave inclusive number of the graph. ▌

Theorem 4.9: Let G be a graph and $e = \{uv\}$ be an edge of G then $e_i(G - v) < e_i(G)$ if and only if $d(u) = \delta(G)$or $d(v) = \delta(G)$.

Proof: Assume that $d(u) = \delta(G)$or $d(v) = \delta(G)$.
Therefore $e_i(G) = \delta(G) + 1$.
Now $d(u)$in $\mathrm{G - e} = \delta(G) - 1$.
$\delta(\mathrm{G - e}) = \delta(G) - 1$
$e_i(G - e) = \delta(G - e) + 1 = \delta(G) - 1 + 1 = \delta(G)$.
Therefore,$e_i(G - e) = \delta(G)$.
$e_i(G - e) = \delta(G) < \delta(G) + 1 = e_i(G)$.
Therefore,$e_i(G - e) < e_i(G)$.
Conversely, suppose $e_i(G - e) < e_i(G)$
Therefore,$\delta(G - e) + 1 < \delta(G) + 1$
Therefore,$\delta(G - e) < \delta(G)$
Therefore,$d(u) = \delta(G)$or $d(v) = \delta(G)$. ▌

Theorem 4.10: Let G be a graph and and $e = \{uv\}$ be an edge of G then $e_i(G - e) = e_i(G)$ if and only if $d(u) \geq \delta(G) + 1$and $d(v) \geq \delta(G) + 1$.

Proof: It is always true that $e_i(G - e) \leq e_i(G)$.
Suppose $d(u) \geq \delta(G) + 1$or $d(v) \geq \delta(G) + 1$.
Then $d(x) \geq \delta(G)$ for all $x$in $G - e$.
Therefore,$e_i(G - e) \geq \delta(G) + 1 = e_i(G)$
Therefore,$e_i(G - e) \geq e_i(G)$
Thus, $e_i(G - e) = e_i(G)$.
Conversely, suppose $e_i(G - e) = e_i(G)$
Therefore,$e_i(G - e) \nless e_i(G)$
Therefore, by theorem 9, $d(u) \geq \delta(G) + 1$and $d(v) \geq \delta(G) + 1$. ▌

Corollary 4.11: Let G be a k_regular graph then for every vertex v of G, $e_i(G - v) < e_i(G)$ also for every edge e, $e_i(G - e) < e_i(G)$.

Proof: (1) If$v \in \mathrm{V(G)}$ then $d(v) = \delta(G) = \mathrm{k}$
Therefore,$e_i(G - v) < e_i(G)$, by theorem 6.
(2) If e is any edge of G then degree of each end vertex of $e$is $k = \delta(G)$.
Therefore, by theorem 10, $e_i(G - e) < e_i(G)$ ▌

Corollary 4.12: Let G be a graph then there are at least $\delta(G)$ edges such that removal of each of them decreases enclave inclusive number of the graph.

Proof: Let v be a vertex of the graph $G$and $d(v) = \delta(G)$.
There are $\delta(G)$ edges whose end vertex is $v$and $d(v) = \delta(G)$.
Therefore by theorem-9,
Removal of any such edge will reduce the enclave inclusive number of the graph.
Thus, there are $\delta(G)$ edges such that removal of each of them reduces the enclave inclusive number of the graph. ▌

V. REFERENCES

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