Closed Support of a Graph under Multiplication

[1]S. Balamurugan , [2]M. Anitha , [3]C. Karnan , [4]P. Palanikumar
[1]PG Department of Mathematics,
Government Arts College, Melur - 625 106, Tamilnadu, India.
[2]Department of Mathematics
Syed Ammal Arts and Science College, Ramanathapuram, Tamilnadu, India.
[3]Department of Mathematics
The Optimus Public School, Erode - 638 455
[4]Department of Mathematics
Mannar Thirumalai Naicker College, Madurai 625 004, Tamilnadu, India.

Abstract

In this paper, the Closed support of a vertex v under multiplication and Closed support of a graph G under multiplication is defined and studied. Also, we find the value of Closed support of some namely graphs like Dutch windmill graph, Butterfly graph and Ladder graph.

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I. INTRODUCTION

In this work we consider finite, undirected, simple graphs $G = (V, E)$ with n vertices and m edges. The *n*eighbourhood of a vertex $v \in V(G)$ is the set $N_G(v)$ of all the vertices adjacent to v in G. For a set $X \subseteq V(G)$, the *open neighbourhood*$N_G(X)$ is defined to be $\cup_{v\in X} N_G(v)$ and the *c*losed neighbourhood $N_G[X] = N_G(X) \cup X$. The *degree* of a vertex $v \in V(G)$ is the number of edges of G incident with v and is denoted by $deg_G(v)$ or $deg(v)$. The maximum and the minimum degrees of the vertices of G are respectively denoted by $\Delta(G)$ and $\delta(G)$. A vertex of a degree 0 in G is called an *isolated* vertex and a vertex of degree 1 is called a *pendent* vertex or an *end* vertex of G. A vertex of a graph G is said to be a vertex of full degree if it is adjacent to all other vertices in G. A graph G is said to be *regular of degree* r if every vertex of G has degree r. Such graphs are called *r-regular* graphs.

The *Dutch windmill graph*$D_n^{(m)}$, is the graph obtained by taking m copies of the cycle graph C_n with a vertex in common. The *Butterfly graph* (also called the bowtie graph and the hourglass graph) is a planar undirected graph with 5 vertices and 6 edges. It can be constructed by joining 2 copies of the cycle graph C_3 with a common vertex. It is denoted by B_n. The *ladder graph*L_n is a planar undirected graph with 2n vertices and n+2(n-1) edges. The Ladder graph obtained as the cartesian product of two graphs one of which has only one edge: $L_{n,1} = P_n \times P_1$.

A closed support of a vertex v under multiplication is defined by $\prod_{u\in N[v]} deg(u)$ and is denoted by $mult[v]$. A closed support of a graph G under multiplication is defined by $\prod_{u\in V(G)} mult[u]$ and it is denoted by $mult[G]$.

II. DEFINITIONS

Definition 2.1.*A closed support of a vertex v under multiplication is defined by $\prod_{u\in N[v]} deg(u)$ and is denoted by $mult[v]$.*

Definition 2.2.*A closed support of a graph G under multiplication is defined by $\prod_{u\in V(G)} mult[u]$ and it is denoted by $mult[G]$.*

III. RESULTS

Proposition 3.1.*For a Path P_n, $(n \geq 2)$, $mult[P_n] = 8^{n-2}$.*
Proof: Let G be a path on n vertices and let $V(G) = \{u_1, u_2, \dots, u_n\}$ where $deg(u_1) = deg(u_n) = 1$ and $deg(u_i) =$

2 for $i = 2,3,\dots,n-1$.

If $n = 2$, 3, or 4, then clearly $mult[G] = 1$, 8 and 64 respectively.

Let $n \geq 5$. Then

$$mult[u_1] = mult[u_n] = 2,$$

$$mult[u_2] = mult[u_{n-1}] = 4$$

and

$$mult[u_i] = \Pi_{v\in N[u_i]} deg(v) = 8.$$

Therefore

$$\begin{aligned} mult[G] &= \Pi_{u\in V(G)} (2^2) \times (4^2) \times (8^{(n-4)}) \\ &= 8^{n-2}. \end{aligned}$$

Theorem 3.2. *For any r-regular connected graph G of order $n \geq 2$, $mult[G] = r^{n(r+1)}$.*

Proof. Let G be a r-regular connected graph on n vertices and let $V(G) = \{u_1, u_2, \dots, u_n\}$ where $deg(u_i) = r$ for all i.

Let $n \geq 2$. Then

$$\begin{aligned} mult[u_i] &= \Pi_{v\in N[u_i]} deg(v) \\ &= r \times r \times \dots \times r \qquad (\mathrm{r+1-times}). \end{aligned}$$

Thus

$$mult[u_i] = r^{r+1}$$

and hence

$$\begin{aligned} mult[G] &= r^{r+1} \times r^{r+1} \times \dots \times r^{r+1} \\ &= r^{n(r+1)}. \end{aligned}$$

Corollary 3.3. *For a Cycle C_n, $(n \geq 3)$, $mult[C_n] = 8^n$.*

Corollary 3.4. *For a complete graph K_n, $(n \geq 2)$, $mult[K_n] = (n-1)^{n^2}$.*

Corollary 3.5. *For a Petersen graph P, $mult[P] = 3^{40}$.*

Proposition 3.6. *For a complete bipartite graph $K_{m,n}$, $(m, n \geq 1)$, $mult[K_{m,n}] = (mn)^{mn} \times n^m \times m^n$.*

Proof. Let $G = K_{m,n}$ be a complete bipartite graph with bipartition (V_1, V_2) where $V_1 = \{u_1, u_2, \dots, u_m\}$ and $V_2 = \{v_1, v_2, \dots, v_n\}$.

Then $deg(u_i) = n$ and $deg(v_j) = m$ for all i, j.

Thus

$$mult[u_i] = \Pi_{v\in N[u_i]} deg(v) = n \times m^n$$

for $i = 1,2,\dots,m$.

Similarly,

$$mult[v_j] = m \times n^m$$

for $j = 1,2,\dots,n$.

Therefore

$$\begin{aligned} mult[G] &= n^m \times m^{nm} \times m^n \times n^{mn} \\ &= (mn)^{mn} \times n^m \times m^n. \end{aligned}$$

Proposition 3.7. *Let $G = L_{2n}$, $(n \geq 4)$, be a Ladder graph. Then $mult[G] = 2^12 \times 3^{8(n-2)}$.*

Proof. Let $G = L_{2n}$, $n \geq 4$. Let $V(G) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ where $deg(u_1) = deg(u_n) = deg(v_1) = deg(v_n) = 2$ and $deg(u_i) = deg(v_i) = 3$ for $i = 2,3,\dots,n-1$.

Then

$$mult[u_1] = mult[v_1] = mult[u_n] = mult[v_n] = 12,$$

$$mult[u_2] = mult[v_2] = mult[u_{n-1}] = mult[v_{n-1}] = 54$$

and

$$mult[u_i] = \Pi_{v\in N[u_i]} deg(v) = 81.$$

Similarly,

$$mult[v_i] = 81.$$

Therefore,

$$\begin{aligned} mult[G] &= \Pi_{u\in V(G)} mult[u] \\ &= (12)^4 \times (54)^4 \times (81)^{2n-8} \\ &= 2^8 \times 3^4 \times 3^8 \times 3^4 \times 2^4 \times 3^{(4n-16)} \times 3^{(4n-16)} \\ &= 2^{12} \times 3^{(8n-16)} \\ &= 2^{12} \times 3^{8(n-2)}. \end{aligned}$$

Proposition 3.8. *For a Fan F_n, $(n \geq 4)$, $mult[F_n] = 2^6 \times 3^{4(n-3)} \times (n-1)^n$.*

Proof: Let $G = F_n (n \geq 4)$. Let $V(G) = \{u, u_1, u_2, \ldots, u_{n-1}\}$ where $deg(u) = n-1$, $deg(u_1) = deg(u_{n-1}) = 2$ and $deg(u_i) = 3$ for $i = 2,3, \ldots, n-2$.

Then

$$mult[u] = (n-1) \times 2^2 \times 3^{n-3},$$

$$mult[u_1] = mult[u_{n-1}] = 6 \times (n-1),$$

$$mult[u_2] = mult[u_{n-2}] = 18 \times (n-1)$$

and

$$mult[u_i] = \Pi_{v\in N[u_i]} deg(v) = 27 \times (n-1)$$

for $i = 2,3, \ldots, n-3$. Therefore

$$\begin{aligned} mult[G] &= \Pi_{v\in V(G)} mult[v] \\ &= 2^2 \times (n-1) \times 3^{n-3} \times 6^2 \times (n-1)^2 \times (18)^2 \times (n-1)^2 \times (27)^{n-5} \times (n-1)^{n-5} \\ &= 2^2 \times (n-1) \times 3^{n-3} \times 2^2 \times 3^2 \times (n-1)^2 \times 2^2 \times 3^4 \times (n-1)^2 \times 3^{3n-15} \times (n-1)^{n-5} \\ &= 2^6 \times 3^{4(n-3)} \times (n-1)^n. \end{aligned}$$

Theorem 3.9 *Let $G = L_{2n}$. Then $mult[G] = 12^4 \times 54^4 \times 81^{2(n-4)}$.*

Proof: Let $G = L_{2n}$ be a Ladder graph with $2n$ vertices.

Let $V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and $deg(v_i) = deg(u_i) = 2$ for all $i = 1, n$; $deg(v_i) = deg(u_i) = 3$ for all $i = 2,3, \ldots, n-1$. Then

$$\begin{aligned} mult[v_1] &= \Pi_{v\in N[v_1]} deg(v) \\ &= deg(u_1) \times deg(v_2) \times deg(v_1) \\ mult[v_1] &= 2 \times 2 \times 3 = 12 \end{aligned}$$

Similarly,

$$mult[v_n] = mult[u_1] = mult[u_n] = 12.$$

$$\begin{aligned} mult[v_2] &= \Pi_{v\in N[v_2]} deg(v) \\ &= deg[u_2] \times deg[v_1] \times deg[v_3] \times deg[v_2] \\ mult[v_1] &= 3 \times 2 \times 3 \times 3 \\ mult[v_2] &= 54 \end{aligned}$$

Similarly,

$$mult[u_2] = mult[u_{n-1}] = mult[v_{n-1}] = 54.$$

Foreach $i = 3,4, \ldots, n-2$,

$$\begin{aligned} mult[v_i] &= \Pi_{v\in N[v_i]} deg(v) \\ &= deg(u_i) + deg(v_{i-1}) + deg(v_{i+1}) + deg(v_i) \\ mult[v_1] &= 3 \times 3 \times 3 \times 3 \\ mult[v_i] &= 81, \ foralli = 3,4, \ldots, n-2. \end{aligned}$$

Similarly,

$$mult[u_i] = 81, foralli = 3,4, \ldots, n-2.$$

Now,

$$\begin{aligned} mult[G] &= \Pi_{v\in V(G)} mult[v] \\ &= \Pi_{i=1,n} mult[v_i] \times \Pi_{i=1,n} mult[u_i] \times \Pi_{i=2,n-1} mult[v_i] \times \Pi_{i=2,n-1} mult[u_i] \\ &\times \Pi_{i=3}^{n-2} mult[v_i] \times \Pi_{i=3}^{n-2} mult[u_i] \\ &= \Pi_{i=1,n} 12 \times \Pi_{i=1,n} 12 \times \Pi_{i=2,n-1} 54 \times \Pi_{i=2,n-1} 54 \times \Pi_{i=3}^{n-2} 81 \times \Pi_{i=3}^{n-2} 81 \\ mult[G] &= 12^4 \times 54^4 \times 81^{2(n-4)}. \end{aligned}$$

Theorem 3.10.*Let* $G = K_m(a_1, a_2, \dots, a_m)$.Then $mult[G] = n^{n(n+1)}$.

Proof. Let $G = K_m(a_1, a_2, \dots, a_m)$ be a multistar graph of order $m + a_1 + a_2 + \cdots + a_m$. Let $V(G) = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n\}$ such that $deg(v_i) = n$and $deg(u_i) = 1$ for all $i = 1,2, \dots, n$.

$$mult[u_i] = \Pi_{v \in N[u_i]} deg(v)$$
$$mult[u_i] = n$$

$$mult[v_i] = \Pi_{v \in N[v_i]} deg(v)$$
$$mult[v_i] = n^n$$

Now,

$$mult[G] = \Pi_{v \in V(G)} mult[v]$$
$$= \Pi_{i=1}^{n} mult[v_i] \times \Pi_{i=1}^{n} mult[u_i]$$
$$mult[G] = n^{n(n+1).}$$

Theorem 3.11.*Let G be a butterfly graph. Then* $mult[G] = 2^{22}$

Proof. Let G be a butterfly graph. Let $V(G) = \{x, v_1, v_2, v_3, v_4\}$ such that $deg(x) = 4; deg(v_i) = 2$,for all $i = 1,2,3,4$;

$$mult[x] = \Pi_{v \in N[x]} deg(v)$$
$$= deg(v_1) \times deg(v_2) \times deg(v_3) \times deg(v_4) \times deg(x)$$
$$= 2 \times 2 \times 2 \times 2 \times 4 = 64$$

For $i = 1,2,3,4$

$$mult[v_i] = \Pi_{v \in N[v_i]} deg(v)$$
$$= 2 \times 4 \times 2$$
$$= 16$$

Now,

$$mult[G] = \Pi_{v \in V(G)} mult[v]$$
$$= mult[v_1] \times mult[v_2] \times mult[v_3] \times mult[v_4] \times mult[x]$$
$$= 2^{22}$$

Theorem 3.12 *Let* $G = D_{(n)}^{(m)}$ *be a Dutch windmill graph.*Then $mult[G] = 2m^{m+1} 2^{3m(n-1)}$

Proof: Let $G = D_{(n)}^{(m)}$ be a Dutch windmill graph. Let $V(G) = \{x, v_1^i, v_2^i, v_3^i, \dots, v_{n-1}^i\}$ for $i = 1,2, \dots, m$ such that $deg(x) = 2m; deg(v_j^i) = 2$,for all $i = 1,2, \dots, m$ and $j = 1,2, \dots, n - 1$;

$$mult[x] = \Pi_{v \in N[x]} deg(v)$$
$$= deg(x) \times \Pi_{i=1}^{m} (deg(v_1^i) \times deg(v_{n-1}^i))$$
$$= 4^m \times 2m$$

For $i = 1,2, \dots, m$

$$mult[v_1^i] = \Pi_{v \in N[v_1^i]} deg(v)$$
$$= 8m$$

Similarly,

$$mult[v_{n-1}^i] = 8m.$$

For$j = 2,3, \dots, n - 2$,

$$mult[v_j^i] = \Pi_{v \in N[v_j^i]} deg(v)$$
$$= 2 \times 2 \times 2 = 8$$

Now,

$$mult[G] = \Pi_{v \in V(G)} mult[v]$$
$$= mult[x] \times \Pi_{i=1}^{m} \Pi_{j=1}^{n-1} mult[v_j^i]$$
$$= 2m^{(m+1)} 2^{3m(n-1)}$$

REFERENCES

[1] M. Anitha, S. Balamurugan: Strong (Weak) Efficient Open Domination on Product Graphs, International Journal of Pure and Applied Mathematics, (J.No. 23425), ISSN 1311 - 8080, communicated.

[2] S. Balamurugan, M. Anitha, C. Karnan and P. Aristotle: *A Note on Open Support of a Graph under Addition I*, Communicated

[3] S. Balamurugan, M. Anitha, C. Karnan and P. Aristotle: *A Note on Open Support of a Graph under Addition II*, Communicated

[4] S. Balamurugan, M. Anitha, C. Karnan and P. Palanikumar: *Closed Support of a Graph Under Addition I* , Communicated

[5] S. Balamurugan, M. Anitha, C. Karnan and P. Palanikumar: *Closed Support of a Graph Under Addition II* , Communicated
[6] S. Balamurugan, M. Anitha, C. Karnan and P. Palanikumar: *Open Support of a Graph Under Multiplication* , Communicated
[7] R. Balakrishnan, K. Ranganathan, *A Textbook of Graph Theory*, Springer,2011.
[8] S. Balamurugan, *A study on chromatic strong domination in graphs*, Ph.D Thesis, Madurai Kamaraj University, India 2008.
[9] J.A. Bondy, U.S.R. Murthy, *Graph theory with Applications*, North-Holland, 1982.
[10] F. Harary : *Graph Theory*, Addsion Wesley, Reading, Mass, (1972).
[11] T. W. Haynes, S. T. Hedetniemi, P. J. Slater: *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, (1998).
[12] C. Y. Ponnappan: *Studies in Graph Theory Support Strong Domination in Graphs*, Ph.D thesis, Madurai Kamaraj University (2008).