Existence of unique integrable solution for a fractional nonlinear Volterra integral equation

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Abstract : We study the existence and uniqueness theorem of a fractional nonlinear Volterra integral equation in the space of Lebesgue integrable on bounded interval [0,T] by using the Banach fixed point theorem.

Keywords : Superposition Operator-Nonlinear Fractional Volterra Integral Equation-Banach Fixed Point Theorem- Lipschitz Condition.

Mathematics Subject Classification : 45M27, 16A05.

I. INTRODUCTION

The subject of nonlinearfractional integral equation considered as an important branch of mathematics because it is used for solving of many problems such as physics, engineering and economics [1,2,3,].
In this paper, we will prove the existence and uniqueness theorem of a fractional Volterra integral equation in the space of Lebesgue integrable$L_1(R^+)$ on bounded interval [0,T]of the kind :

$$x(t) = g(t)\,f(t, x(t)) + h(t) +$$

$$+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, f\big(s, x(s)\big)ds, 0 < \alpha < 1, \quad t \in [0,T] \quad (1.1)$$

II. PRELIMINARIES

Let Rbe the field of real number, R^+be the interval $[0,\infty]$. If $A$is a Lebesgue measurable subset of R,then the symbol mas(A) stands for theLebesgue measure of A.
Further, denoted by $L_1(A)$ the space of all real function, defined and Lebesgue measurable on the set A. The norm of a function $x \,\varepsilon\, L_1(A)$ is define in the standard way by the formula,

$$\|x\| = \|L_1(A)\| = \int_A |x(t)|\,dt$$

Obviously $L_1(A)$ forms a Banach space under this norm, the space $L_1(A)$ will be called the Lebesgue space. In the case when $A = R^+$we will write L_1 instead of $L_1(R^+)$.
One of the most important operator studied in nonlinear functional analysis is the so called the superposition operator [4]. Now, let us assume that $A \subset R$ is a given interval bounded.

Definition 2.1[4]**:** Assume that a function $f(t,x) = f\colon I \times R \to R$satisfies the so-called Carathéodory condition, i. e. it is measurable in t for any $x \in R$ and continuous in x for almost all $t \in I$. Then to every function $x = x(t)$ which is measurable on I we may assign the function $(Fx)(t) = f\big(t, x(t)\big)$, $t \in I$. The operator F defined in such a way is said to be the **superpositionoperator** generated by the function f.

Theorem 2.1 [5].

The superposition operator F generated by a function f maps continuously the space $L^1(I)$ into itself if and only if $|f(t,x)| \le a(t) + b|x|$ for all $t \in I$ and $x \in R$, where $a(t)$is a function the from $L^1(I)$ and b is a nonnegative constant.

This theorem was proved by Krasnoselskii in the case when I is bounded interval. The generalization to the case of an unbounded interval I was given by Appell and Zabrejko[4].

Definition 2.2 [6]:A function $f : A \to R^m$, $A \subset R^n$, is said to be Lipschitz continuous if there exists a constant L, L > 0 (is called the Lipschitz constant of f on A) such that

$$|f(x) - f(y)| \le L\, |x - y|, \text{for all } x, y \in A.$$

Definition 2.3 [7]Let (X,d) be a metric space and $T:X\to X$ is called contraction mapping, if there exist a number $\gamma<1$, such that : $d(Tx,Ty)\le\gamma\, d(x,y),\quad \forall\, x,y\in X.$

Theorem 2.2 [8]: Let X be a closed subset of a Banach space E and $T:X\to X$ be a cont-raction, then T has a unique fixed point.

Definition 2.4[9]:Let $[a,b](-\infty<a<b<\infty)$ be a finite interval on the real axis R, the Riemann-Liouville fractional integral $I^{\alpha}_{a^+}f$ of order $\alpha\in C(\mathcal{R}(\alpha)>0)$ is define by :

$$I^{\alpha}_{a^+}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\frac{f(s)ds}{(t-s)^{1-\alpha}}\qquad (\ t>a\,;\ \mathcal{R}(\alpha)>0\ ).$$

Definition 2.5[10] :Let $[a,b](-\infty<a<b<\infty)$ be a finite interval on the real axis R, the Riemann-Liouville fractional integral $I^{\alpha}_{a^+}f$ of order $\alpha\in C(\mathcal{R}(\alpha)\ge 0)$ is define by :

$$D^{\alpha}_{a^+}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_a^t\frac{f(s)ds}{(t-s)^{1-n+\alpha}}\quad (t>a\,;n=[\mathcal{R}(\alpha)]+1)$$

Where $[\mathcal{R}(\alpha)]$denotes the integral part of $\mathcal{R}(\alpha)$. " i.e. $[\mathcal{R}(\alpha)]$ satisfies

$[\mathcal{R}(\alpha)]\ \le\mathcal{R}(\alpha)\ \le[\mathcal{R}(\alpha)]+1.$ "

III. EXISTENCE THEOREM

Define the operator H associated with integral equation (1.1) take the following form.

$Hx=Ax+Bx.$ (3.1)

Where

$(Ax)(t)\ =g(t)\,f(t,x(t)),$

$(Bx)(t)=h(t)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f\big(s,x(s)\big)ds$

$=h(t)+KFx(t),$

Where,$(Kx)(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)\,ds,$

$Fx=f(t,x)$, are linear operator at superposition respectively.

We shall treat the equation (3.1) under the following assumptions listed below.

Assume that :

i) g : $[0,T]\to R$ is bounded function such that : $M=\sup_{t\in[0,T]}|g(t)|,$

and $h:R^+\to R$, such that $h\in L_1[0,T]$.

ii) $f:[0,T]\times R\to R$ satisfiesLipschitz condition with positive constant$L$ such that ;

$|f(t,x(t))-f(t,y(t))|\ \le L|x(t)-y(t)|,$ for all$t\in[0,T]$.

iii) $LM+\frac{LT^{\alpha}}{\Gamma(\alpha+1)}<1.$

Now, for the existence of a unique solution of our equation, we can see the following theorem .

Theorem 3.1 : If the assumptions (i)-(iii) are satisfied, then the equation (1.1) has a unique solution, where $x\in L_1[0,T]$.

Proof : first we will prove that $H:L_1[0,T]\to L_1[0,T]$,

second will prove that H is contraction .

Consider the operator H as :

$$Hx(t)=g(t)\,f\big(t,x(t)\big)+h(t)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f\big(s,x(s)\big)ds$$

Then our equation (1.1) becomes

$x(t)=Hx(t).$

We notice that by assumption (ii), we have

$$|f(t,x)| = |f(t,x) - f(t,0) + f(t,0)|$$

$$\leq |f(t,x) - f(t,0)| + |f(t,0)|$$

$$\leq L\,|x - 0| + |f(t,0)|$$

$$\leq L\,|x| + a(t)$$

Where

$$|f(t,0)| = a(t)$$

To prove that $H : L_1\,[0,T] \to L_1\,[0,T]$,

let $x \in L_1[0,T]$,

then we have

$$\|Hx(t)\| = \int_0^T |(Hx)(t)|dt$$

$$\leq \int_0^T \left| g(t)\, f\big(t,x(t)\big) + h(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, f\big(s,x(s)\big)ds \right| dt$$

$$\leq \int_0^T |g(t)|\left| f\big(t,x(t)\big)\right|dt + \int_0^T \left| h(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, f\big(s,x(s)\big)ds \right| dt$$

$$\leq M \int_0^T [a(t) + L|x(t)|]dt + \int_0^T |h(t)|dt + \int_0^T \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, f\big(s,x(s)\big)ds \right| dt$$

$$\leq M\,\|a\| + ML \int_0^T |x(t)|dt + \int_0^T |h(t)|dt + \int_0^T \int_s^T \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|f\big(s,x(s)\big)dt\right|ds$$

$$\leq M\,\|a\| + ML \int_0^T |x(t)|dt + \|h\| + \frac{(t-s)^{\alpha}}{\alpha\Gamma(\alpha)}\Big|_s^T \int_0^T \left|f\big(s,x(s)\big)\right|ds$$

$$\leq M\,\|a\| + ML \int_0^T |x(t)|dt + \|h\| + \frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)} \int_0^T [a(s) + |x(s)|]ds$$

$$\leq M\,\|a\| + ML \int_0^T |x(t)|dt + \|h\| + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\|a\| + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\|x\|$$

$$\leq \left[M + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \right] \|a\| + \|h\| + \left[ML + L\frac{T^{\alpha}}{\Gamma(\alpha+1)} \right] \int_0^T |x(t)|dt < \infty$$

Then

$H : L_1\,[0,T] \to L_1\,[0,T]$

second to prove that H is contraction,

let $x, y \in L_1\,[0,T]$, then

$$\int_0^T |Hx(t) - Hy(t)|dt = \int_0^T |\, g(t)\, f\big(t,x(t)\big) + h(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, f\big(s,x(s)\big)ds$$

$$- g(t)\, f\big(t,y(t)\big) - h(t) - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, f\big(s,y(s)\big)ds\, |dt$$

$$\leq \int_0^T |g(t)| \left|f\big(t,x(t)\big) - f\big(t,y(t)\big)\right|dt$$

$$+\int_0^T\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f(s,x(s))ds - f(s,y(s))\right|dsdt$$

$$\leq M\int_0^T L|x(t)-y(t)|dt$$

$$+\int_0^T\int_s^T \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f(s,x(s))ds - f(s,y(s))\right|dtds$$

$$\leq LM\,\|x-y\| + \frac{(t-s)^{\alpha}}{\alpha\Gamma(\alpha)}\Big|_s^T\int_0^T\left|f(s,x(s))ds - f(s,y(s))\right|ds$$

$$\leq\ LM\,\|x-y\| + \frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}\int_0^T L|x(s)-y(s)|ds$$

$$\leq\ LM\,\|x-y\| + \frac{L\,T^{\alpha}}{\Gamma(\alpha+1)}\int_0^T |x(s)-y(s)|ds$$

$$\leq\ LM\,\|x-y\| + \frac{L\,T^{\alpha}}{\Gamma(\alpha+1)}\|x-y\|$$

$$\leq [LM\ + \frac{LT^{\alpha}}{\Gamma(\alpha+1)}]\|x-y\|$$

Hence, by using Banach fixed point theorem,
H has a unique point, which is the solution of the equation (2.2), where $x\,\epsilon\,L_1\,[0,T]$. ■

CONCLUSION

in this paper, by using Banach fixed point theorem we proved the existence and uniqueness theoremof a fractional nonlinear Volterra integral equation in the space of Lebesgue integrable on bounded interval [0,T]of the kind :

$x(t) = g(t)\,f(t,x(t)) + h(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s,x(s)\right)ds, 0 < \alpha < 1,\ t \in [0,T]$under the following assumptions :

i) g : $[0,T] \rightarrow R$ is bounded function such that : $M = \sup_{t\in[0,T]} |g(t)|$,

and $h : R^{+} \rightarrow R$, such that $h \in L_1\,[0,T]$.

ii) $f : [0,T] \times R \rightarrow R$ satisfiesLipschitz condition with positive constant$L$ such that ;

$|f(t,x(t)) - f(t,y(t))| \leq L|x(t)-y(t)|$, for all$t \in [0,T]$.

iii) $LM + \frac{LT^{\alpha}}{\Gamma(\alpha+1)} < 1$.

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