Hypersurface Homogeneous Cosmological Models with Stiff-Matter in General Relativity

Dr. Mantu Kumar
P.G. Department of Mathematics
Veer kunwar Singh University, Ara-802301

***Abstract** : In this paper, we present Einstein's field equations for perfect fluid within the frame work of hyper surface homogeneous space-time. Exact solutions of the field equations are obtained which represent cosmological models in the presence of stiff fluid. The physical and kenematical behaviors of the cosmological models are discussed.*

Keyword: *Hypersurface homogeneous space-time. Stiff-matter cosmological models.*

I. INTRODUCTION

Astronomical observations indicate that on large scales the present day universe is both homogeneous and isotropic, the latter having been confirmed by the discovery of 2.7 ^{0}K microwave background radiation [3,12,13]. The simplest spatially homogeneous space-time is given by Robertson-Walker metric. It is based on the following assumptions.

(i) There exists a global time coordinate which serves as the timecoordinate of a Gaussian coordinate system.
(ii) The three dimensional space t = constant are locally isotropic.
(iii) Any two points in a three-space belonging to a given fixed time are equivalent.

Homogeneous isotropic models form the simplest class of cosmological models. The observational evidence shows that the anisotropy is negligible in the large scale picture of the universe but it is not so at smaller scale. The universe in early days did not have the same property of isotropy that is found at present. According to the group theoretic criterion the spatially homogeneous space-times are of two types:

(i) Bianchi space-times, which posses a three-dimensional group of isometriesG_3 acting simply transitively on the tree-dimensional space like hypersurfaces.
(ii) Kantowaski-Sachs metric which a four dimensional group of isometriesG_4 multiply transitive on these space sections.

Friedmann-Robertson-Walker (FRW) models [6,7,15,16,28,29] which are both spatially homogeneous and isotropic admit six-parameter group of isometries. Each of these groups contains a particular three-parameter group G_3 as a subgroup. The spatially homogeneous Bianchi-spaces I – IX are useful tools for constructing cosmological models to describe the behaviors of the universe at early stages of its evolution [19]. Bianchi – VI_0 spaces are of particular interest since they are sufficiently complex, while at the same time, they are a simple generalization of Bianchi–I spaces. Since long a great deal of theoretical work has been done to build Bianchi-VI_0 cosmological models by solving Einstein's field equations associated with different matter distributions. Ellis and MacCallum [5] obtained solutions of Einstein's field equation in the case of a stiff-fluid. Collins [2] and Ruban [18] presented some exact solutions of this type for perfect fluid distribution satisfying the specific equations of state. Dunn and Tupper [4] investigated a class of Bianchi type VI_0 perfect fluid cosmological model associated with electromagnetic fields. Lorentz [10] generalized the dust model given by Ellis and MacCallum [5]. Roy and Singh [17] derived some exact solutions of Einstein-Maxwell equations representing a free gravitational field of the magnetic type with perfect fluid and incident magnetic field. Ribeiro and Sanyal [14] studied spatially homogeneous Bianchi – VI_0 models containing a viscous fluid in the presence of an axial magnetic field. Following Hajj-Boutrös [8], [9].Shri Ram [20] presented an algorithm for generating a new exact perfect fluid solutions of Einstein's field equations for spatially homogeneous cosmological models of Bianchi type VI_0 without choosing any equation of state.

Stewart and Ellis [25] obtained general solution of Einstein field equation in the case of hypersurface orthogonal space-times with metric,

$$ds^2 = -dt^2 + A^2 (t)\, dx^2 + B^2 (t)\, (dy^2 + \textstyle\sum^2 (y, k)\, dz^2) \qquad (1.1)$$

where

$$\sum(y,k) = \begin{cases} Sin\ y & ; k = 1 \\ y & ; k = 0 \\ Sinh\ y & ; k = -1 \end{cases}$$

The perfect fluid filling the space-time satisfies the equation of state

$$p = (\gamma - 1)\rho \qquad (1.2)$$

where p is the pressure, ρ is the energy density and γ is a parameter satisfying $1 < \gamma < 2$

The metric (1.1), admits a group of motion G_4 in V_3 the isotropy group being spatial rotations. Hajj-Boutrös [9] presented a method to generate exact solutions of the field equations in the case of hypersurface-homogeneous space-times filled with a perfect fluid not satisfying the equation of state (1.2). Recently,Mazumdar [11] considered LRSBianch–1 space-times filled with a perfect fluid and has shown that the field equations are solvable for any arbitrary cosmic scale function.

Latter onVerma and Shri Ram [27] have investigated some hyersurface-homogeneous cosmological models with bulk viscous fluid and time -dependent cosmological term. In sequel, Shri Ram and Verma [22] have obtained a bulk viscous fluid cosmological model with time-varying gravitational constant and cosmological term. Within the framework as hypersurface-homogeneous space-time.Chandel et al. [1] have studied hypersurface- homogeneous bulk viscous fluid models with decaying cosmological term. In the context of present-day accelerating model as the universe, Shri Ram and Chandel [21] have investigated hypersurface-homogeneous cosmological models with matter and dark energy. Shri Ram et al. [23] have presented hypersurface -homogeneous cosmological models with dynamical equation of state parameter in Lyra geometry. Further, Shri Ram et al. [24] have invertigated hyper-surface homogeneous cosmological models with anisotropic dark energy in Saxz-Ballester theory of gravity. Recently, Tiwari and Mishra [26] have obtained hypersurface-homogeneous bulk viscous fluid cosmological models with time-dependent gravitational constant and cosmological term following the technique developed by Mazumdar [11].

In this paper, we obtain the solutions of Einstein's filed equations in the presence of a stiff-matter which represent hypersurface-homogeneous cosmological models. We also discuss the physical and kinematical behaviors of the models of the models presented.

II. Field equations

Einstein's field equations in the presence of a perfect fluid are

$$R_{ij} - \frac{1}{2}Rg_{ij} = -T_{ij} \qquad (2.1)$$

whereR_{ij} is the Ricci tensor, R is scalar curvature and T_{ij} is the energy-momentum tensor of a perfect fluid given by

$$Tij = (\rho + p)\, v_i.v_j = pg_{ij}$$

ρ being the density, p the pressure and v^i is the 4-velocity satisfying

$$v_i v^i = 1$$

For a Ricci-tensor of type [(1,1,1),1] in segre notation, the Einstein field equations (2.1) for a perfect fluid distribution are

$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{k}{B^2} = -p \qquad (2.2)$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} = -p \qquad (2.3)$$

$$2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{k}{B^2} = -\rho \qquad (2.4)$$

Where a dot denotes differentiation with respect to 't'

III. Stiff-Matter Case

We now assume the case of stiff-matter for which

$$\rho = p \qquad (3.1)$$

The possible relevance of this equation of state as regards the matter content of the universe in its early stages has been discussed by a number of authors.

For stiff matter, equations (3.1) and (2.4) provide

$$\frac{\ddot{B}}{B}+\frac{\dot{B}^2}{B^2}+\frac{\dot{A}\dot{B}}{AB}+\frac{k}{B^2}=0 \qquad (3.2)$$

Thus, the independent field equations to be solved are

$$\frac{\dot{B}^2}{B^2}+2\frac{\dot{A}\dot{B}}{AB}+\frac{k}{B^2}=\rho \qquad (3.3)$$

$$\frac{\ddot{B}}{B}+\left(\frac{\dot{B}}{B}\right)^2+\frac{\dot{A}\dot{B}}{AB}+\frac{k}{B^2}=0 \qquad (3.4)$$

$$\frac{\ddot{A}}{A}+\frac{2\dot{A}\dot{B}}{AB}=0 \qquad (3.5)$$

To treat these equations, we introduce the new variable τ by

$$dt = AB^2 d\tau \qquad (3.6)$$

The equation (3.5) reduce to

$$\frac{A''}{A}-\left(\frac{A'}{A}\right)^2=0 \qquad (3.7)$$

where a dash stands for $d/d\tau$. Integrating (3.7) we obtain

$$A = K_1 e^{a\tau} \qquad (3.8)$$

where a and $K_1$are arbitray constants, without loss of generality, we can take K_1=1 Thus.

$$A = e^{a\tau} \qquad (3.9)$$

Equation (3.4) can be written as

$$(\log B)'' + k(AB)^2 = 0 \qquad (3.10)$$

Setting$B = e^{\alpha}$ and using (3.9), equation (3.10) reduces to

$$\alpha'' + ke^{2(a\tau+\alpha)} = 0 \qquad (3.11)$$

which can also be written

$$X'' + ke^{2x} = 0 \qquad (3.12)$$

where $X = a\tau + \alpha$ Solving (3.12), we obtain

$$\tau+\tau_0 = \pm\int\frac{dX}{\sqrt{c^2-ke^{2X}}} \qquad (3.13)$$

where c and τ_0 are integration constants. Again, putting e^{-X} = Z, the equation (3.13) can be written as

$$\tau+\tau_0 = \mp\frac{1}{c}\int\frac{dZ}{\sqrt{Z^2-k/c^2}} \qquad (3.14)$$

Considering minus sign before the integral on the right hand side of (3.14), we now present some physically meaningful solution for k=0, 1 and -1.

3.1 when k = 0

Equation (3.14) yields

$$c(\tau+\tau_0) = -\int\frac{dZ}{Z} \qquad (3.15)$$

Integrating (3.15), we obtain

$$Z^{-1} = e^{c(\tau+\tau_0)} \qquad (3.16)$$

From equations (3.13) and (3.16), we get

$$X = c(\tau+\tau_0) \qquad (3.17)$$

Again form equations (3.12) and (3.17), we obtain

$$\alpha = (c - a)\tau + \tau_0 \quad (3.18)$$

Using (3.18) in equation (3.10), we finally obtain

$$(B = K_2 e^{(c-a)\tau} \quad (3.19)$$

Without loss of generality we can assume $K_2 = 1$, Hence,

$$B = e^{(c-a)\tau} \quad (3.20)$$

The metric of the solutions can be written in the form

$$ds^2 = -dt^2 + e^{2a\tau}dx^2 + e^{2(c-a)\tau}(dy^2 + y^2dz^2) \quad (3.21)$$

For the model (3.21) pressure and energy density are given by

$$p = \rho\,(c^2 - a^2)e^{(a-2c)\tau} \quad (3.22)$$

The physical and kinematical quantities of the model (3.21) have the following expressions.

Spatial Volume (V^3):

$$V^3 = e^{(2c-a)\tau} \quad (3.23)$$

Expansion Scalar (θ) :

$$\theta = (2c - a)e^{(a-2c)\tau} \quad (3.24)$$

Shear Scalar (σ) :

$$\sigma = -\frac{c}{\sqrt{3}}e^{(a-2c)\tau} \quad (3.25)$$

Deceleration parameter (q):

$$q = -2 \quad (3.26)$$

The energy-density and pressure are positive if c > a. For the physical reality of the model (3.21), we also take a > 0. The energy density, expansion scalar and sher scalar are decreasing function of τwhich tend to zero as $\tau \to \propto$. The spatial volume in infinite as $\tau \to \propto$. The model starts expanding from finite volume as $\tau = 0$ and gives essentially an empty space-time for large τ. The negative value of q = -2 shows super inflation in the univers.

3.2 when K = 1

Equation (3.14) becomes

$$\tau + \tau_0 = -\frac{1}{c}\int \frac{dZ}{\sqrt{Z^2 \frac{1}{c^2}}} \quad (3.27)$$

Integrating equation (3.27), we obtain

$$Z^{-1} = c\, sech\{c(\tau + \tau_0)\} \quad (3.28)$$

From equations (3.13) and (3.28), we get

$$X = \log\,[c\ \mathrm{sech}\ \{c(\tau + \tau_0)\}] \quad (3.29)$$

Again, from equations (3.12) and (3.29), we obtain

$$\alpha = \log[c\,\mathrm{sech}\{c(\tau + \tau_0)\}] - a\tau \quad (3.30)$$

Using (3.30), equation (3.10), yields

$$B = ce^{-a\tau}\ \mathrm{sech}\{c(\tau + \tau_0)\} \quad (3.31)$$

Without loss of any generality we can take $\tau_0 = 0$

The metric of the solution can be written as

$$\mathrm{ds}^2 = -\mathrm{dt}^2 + \mathrm{e}^{2a\tau}\mathrm{dx}^2 + c^2\mathrm{e}^{2a\tau}\,\mathrm{sech}(c\tau)\,[dy^2 + sin^2 y\, dz^2 \tag{3.32}$$

For the model (3.32), pressure and energy density are given by

$$p = \rho = \frac{(c^2-a^2)}{c^4} e^{2a\tau} cosh^4(c\tau) \tag{3.33}$$

Without are nonnegative if c > a > 0

The physical and kinematical quantities for the model (3.32) have the following expressions.

Spatial Volume (V^3):

$$V^3 = c^2 e^{-a\tau} sech^2(c\tau) \tag{3.34}$$

Expansion Scalar (θ) :

$$\theta = \frac{-[a+2ctan\,h(c\tau)]}{[c^2 e^{-a\tau} sec\,h^2(c\tau)]} \tag{3.35}$$

Shear Scalar (σ):

$$\sigma = \frac{1}{\sqrt{3}} \frac{[2a+c\,Tan\,h\,(c\tau)]}{c^2 e^{-az} sec\,h^2(c\tau)} \tag{3.36}$$

Deceleration parameter (q):

$$q = \frac{6c^2[2tan\,h^2(c\tau)-tan\,h(c\tau)-2]}{[a+2ctan\,h(c\tau)]^2} + 1 \tag{3.37}$$

The model (3.32) has no finite singularity. The spatial volume tends to zero and the energy density tend to infinity a $\tau \to \propto$.The scalar expansion is always negative. Thus, the space-time (3.32) is a contracting model of the universe.

3.3 when k = **-**1

Equation (3.14) becomes

$$\tau + \tau_0 = -\frac{1}{c}\int \frac{dZ}{\sqrt{Z^2+\frac{1}{c^2}}} \tag{3.38}$$

Without loss of generality we can assume $\tau_0 = 0$.On integration equation (3.38) yields.

$$Z^{-1} = c\cos ech(c\tau) \tag{3.39}$$

From equations (3.31) and (3.39), we obtain

$$X = \log\{c\ cosech\ (c\tau)\} \tag{3.40}$$

Again, form equations (3.40) and (3.12), we obtain

$$\alpha = \log\{c\ cosech\ (c\tau)\} - a\tau \tag{3.41}$$

Using (3.41), equation (3.10), becomes

$$B = c\ e^{-a\tau}\ cosech\ (c\tau) \tag{3.42}$$

The metric of the solutions can be written as

$$\mathrm{ds}^2 = -\mathrm{dt}^2 + \mathrm{e}^{2a\tau}\mathrm{dx}^2 + c^2\mathrm{e}^{-2a\tau}\,\mathrm{cosech}(c\tau)\,[dy^2 + sinh^2 y\, dz^2 \tag{3.43}$$

Pressure and energy density for the model (3.43) are given by

$$\rho = \rho = \frac{(C^2 a^2)}{c^4} e^{2a\tau} sinh^4\,(c\tau) \tag{3.44}$$

For the model (3.43), the physical and kinematical quantities have the following expressions.

Spatial Volume (V^3):

$$V^3 = c^2 e^{-a\tau} cosech^2(c\tau) \tag{3.45}$$

Expansion Scalar (θ):

$$\theta = -\frac{e^{a\tau}}{c^2}\{a + 2c\coth(c\tau)\}sinh^2(c\tau) \tag{3.46}$$

Shear Scalar (σ):

$$\sigma = \frac{e^{at}}{c^2\sqrt{3}}\{2a + c\coth(c\tau)\}sinh^2\,(c\tau) \tag{3.47}$$

Deceleration parameter (p) :

$$q\ \frac{3[-2\{a+c\coth(c\tau)]^2 + a^2 - 2c^2]}{[a+2c\coth(c\tau)]^2} + 1 \tag{3.48}$$

The model (3.43) has no finite singularity. The spatial volume tends to zero and the energy density tend to infinity as $\tau \to \propto$. The scalar expansion is always negative. Thus, the space-time (3.43) is also a contracting model of the universe filled with stiff matter.

IV. CONCLUSION

We have investigated hypersurface-homogeneous cosmological models counting stiff -matter. Exact solutions of field equations have been obtained which correspond to accelerating / decelerating models of the universe. Expressions for some important cosmological parameters have been obtained and physical behaviors of the models are discussed in detail. These models are certainly useful in the study of dynamical behaviors of homogeneous and anisotropic cosmological models.

V. ACKNOWLEDGEMENTS

The author is grateful to the reviewer for his constructive comments which have considerably improved the presentation of the paper and also thank full to Veer Kunwar Singh University for providing all the infrastructure of my research work.

REFERENCES

[1]. Chandel, S, singhM.K. and Shri Ram (2012) : Ad. studies, theorphys (2012)., 6, 1189.
[2]. Collins, C.B. (1971): Comm. Math. Phys., 23, 137.
[3]. Conklin, E.K. and Bracewell, R.N. (1967): Phys. Rev. Letters, 18, 614.
[4]. Dunn, K.A. and Tupper, B.O.J. (1976): Astrophys. J., 180, 317.
[5]. Ellis, G.F.R. and MacCallum, M.A.H. (1969): Comm. Math. Phys., 12, 20.
[6]. Friedmann, A. (1922): Z. Physik, 10, 377.
[7]. Friedmann, A. (1924): Z. Physik, 21, 326.
[8]. Hajj-Boutrös, J. (1984): In Gravitation Geometry and Relativity Physics, Springer, Berlin, p-51.
[9]. Hajj-Boutrös, J. (1985): J. Math. Phys., 26, 2297
[10]. Lorentz, D. (1982): Astrophys. Space Sci., 85, 69.
[11]. Mazumdar, A. (1994): Ger. Rel. Grav., 26, 307.
[12]. Partridge, R.B. and Wilkinson, D.T. (1967): Phys. Rev. Letters, 18, 557.
[13]. Penzias, A.A. and Wilson, R.W. (1967): Science, 156, 1100.
[14]. Ribeiro, B.M. and Sanyal, A.K. (1987): J. Math. Phys., 28, 657.
[15]. Robertson, H.P. (1935): Astrophys. J., 82, 284.
[16]. Robertson, H.P. (1936): Astrophys. J., 83, 187, 256.
[17]. Roy, S.R. and Singh, J.P. (1983): ActaPhysicaAustriaca, 5, 57.
[18]. Ruban, V.A. (1978): Preprint no. 412, Leningred Institute of Nuclear Physics B.P. Konstrantinova.
[19]. Ryan, M. and Shepley, L.C. (1975): Homogeneous Relativistic Cosmologies, Princeton University Press, Princeton, newJersy.
[20]. Shri Ram (1989): Int. J. Theor. Phys., 28, 917.
[21]. Shri Ram and Chandel, S (2014)., : Journal of dynamical systems with Geometric theories (2014) :12, . 93.
[22]. Shri Ram and Verma, M.K. (2010): Astrophys. Space sci., 330, 157.
[23]. Shri Ram, Chandel, S and Verma, M.K. (2015): Candian journal of phys. (2015): 93,1009 (2015).
[24]. Shri Ram, Chandel, S and Verma, M.KPramana : J. Phys., (2016): 88, 55.
[25]. Stewart, J. M. and Ellis, G.F.R. (1968): J. Math Phys., 9, 1072.
[26]. Tiwari, R.K. and Mishra, V. (2018) :7, 2111.
[27]. Verma ,M.K. and Shri Ram (2010) : Astrophys. space science., 326, 299.
[28]. Walker, A.G. (1935): Quart. J. Math. Oxford Ser, 6, 81.
[29]. Walker, A.G. (1936): Proc. Lond. Math. Soc. (2), 42, 90.