Wheel Planar Graphs in Topology

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Abstract :*This paper purports to explain an area of planar graphs in topology. This paper highlights the unique qualities of wheel in connecting with planar graphs. That is, a vehicle rolls smoothly over a surface when the axle is placed at the centre of a circular wheel. This paper is an attempt to study a brief structure in topological wheels.*

Keywords : *Wheels – Wheel graph-Topological Wheel.*

INTRODUCTION

In our day to day life, wheels play a very important role. For instance it plays an important role in transportation. A wheel is a graph formed by a cycle and a vertex that has atleast three neighbours. Kuratowski, Thomassen and Tutte says the characterization of planar wheels in topology.

Wheels

Before the invention of the wheel, some cultures used cylindrical rollers when moving heavy loads. The load was rolled along over the cylinders, with the disadvantage that they had to be continually replaced under the front.

About 5500 years ago, an anonymous Sumerian in Mesopotamia invented what must surely be mankind's single greatest technological achievement-the wheel.

Nowadays, wheels play a very important role in everyday life. From agriculture to the exploration of Mars, from people transportation to massive movement of products; everything is possible due to the existence of the wheel.

When included as components of a vehicle, wheels allow the vehicle to roll smoothly over a surface. The wheel is round because a circle is the geometric locus of points equidistant from a fixed point. An axle placed at the centre of the wheel will stay at a constant altitude from the ground as the wheel rotates.

Most people think the circle is the only shape wheels can have.

Different figures rolling over modified surfaces can be found in science exhibitions around the world. If we use rollers, rather than wheels on axles fixed to the vehicle, then any constant-breadth shape will do in place of the circle.

Obviously a wheel must be made in the form of a circle with the hub at the centre, since any other form will produce an up-and-down motion.

In mathematical background we must begin with convex sets, their representation by a support function, and some special convex sets.

A set of points K is convex if it contains every line segment with end-points in K. If in addition, K is bounded and has interior points, its boundary (denoted by δK) is called a closed convex curve.

A line L is a support line of the convex set K at point $a \in \delta K$ if it has the following properties.

- $a \in L$
- K is contained in the closure of one of the two open half-planes into which L cuts the plane

Note that every point on δK lies on a support line and there are exactly two support lines perpendicular to each direction.

We can represent K by the set of its support lines. A support line L may be parametrized by (q, p) where q is the angle between its normal and the x-axis and p is the distance from L to a fixed interior point of K. Since p is uniquely determined by q, the set of support lines is $\{(q, p(q)) : q \in [0, 2\pi)\}$. The support function of K is p(q).

For any convex set K, the union of all closed disks of radius r and centres in K, denoted by K_r, has boundary δK and any $y \in \delta K_r$.

$r = \min\{dist(x, z) : z \in \delta K_r\}$

$\;\;= \min\{dist(y, z) : z \in \delta K\}$.

Clearly, if p (q) is the support function of K, then the support function of K_r is p(q)+r.

The distance between the two parallel support lines of a convex set K that are perpendicular to the direction q is the breadth b(q) of K. We have

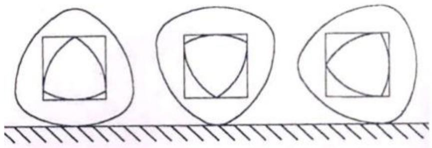
$b(q)=p(q)+p(q+\pi)$

If b(q) is the same for all q, the set K is said to be of constant breadth.

To employ non-circular wheels we need to ensure that when the vehicle moves, it maintains the same distance from the ground.

Consider the following structure:

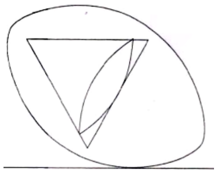
- A square attached rigidly to the vehicle (with two of its sides parallel to the ground).
- A constant-breadth axle that rotates inside the square, maintaining contact with its sides.
- A wheel that is attached rigidly to the axle.



The movement of a constant breadth wheel

For triangular-based wheels, consider the following structure:

- An equilateral triangle rigidly attached to the vehicle (with its upper side parallel to the ground).
- A triangular set as the axle that rotates inside the triangle, maintaining contact with its sides.
- A wheel that is attached rigidly to the axle.



A triangular wheel

Generalizing, we can attach any convex polygon P to the vehicle, have an axle A that rotates inside (touching at all times the sides of the polygon), and seek a wheel such that, when it rotates, the polygon remains at constant altitude from the ground.

Since every convex polygon possesses at least one of the following properties:

- it is a parallelogram
- the extension of three of its sides forms a triangle that contains the polygon,

we can focus on axles that rotate inside parallelograms and triangles.

An axle that rotates inside a parallelogram has constant breadth; therefore its support function $a(\theta)$ satisfies $a(\theta)+a(\theta+\pi)=\text{constant}$.

For every triangle there exists at least one figure that can rotate inside.

It is clear that at whatever speed a circular wheel moves, it will always look around. For non-circular wheels, the path that is traced (relative to the vehicle) by a point on the wheel is no longer a circle. Therefore, the wheel while rotating may not look round.

Similarly, if the wheels are triangular sets rotating inside a triangle, then the path traced resembles a triangle with rounded corners [4].

Wheel Graph

A graph G contains a graph F if an induced subgraph of G is isomorphic to F. An element of a graph is a vertex or an edge. When S is a set of elements of G, we denote by G\S the graph obtained from G by deleting all edges of S and all vertices of S.

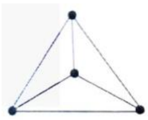
A wheel is a graph formed by a chordless cycle C and a vertex u not in C that has at least three neighbors in C. Such a wheel is denoted by (u, C); u is the center of the wheel and C the rim. Observe that K_4 is a wheel. Chudnovsky states that every non-null wheel-free graph contains a vertex whose neighborhood is made of disjoint cliques with no edges between them.

The class of graphs that do not contain a subdivision of a wheel as an induced subgraph is the class of graphs that do not contain a wheel or a subdivisions of K_4 as induced subgraphs.

A cycle C_n is a graph of n vertices with $V(C_n)=\{v_1, v_2, \ldots, v_n\}$such that $E(C_n)=\{v_iv_{i+1} \mid 1 \le i \le n\text{-}1\} \cup \{v_1v_n\}$

For $n \ge 4$, a wheel W_n is defined to be a graph of n vertices with

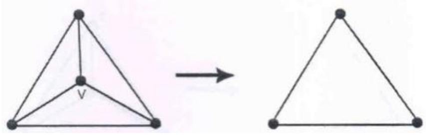
$V(W_n)=V(C_{n-1}) \cup \{uv \mid u=V(K_1)\}$ and $v \in V(C_{n-1})$



K_4 or W_4

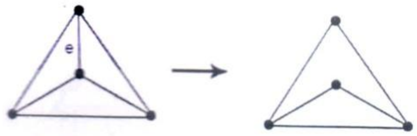
Let G be a graph, $v \in V(G)$ and $e = uv \in E(G)$.

The subgraph G-v with $V(G\text{-}v)=V(G)\backslash v$ and $E(G\text{-}v)=E(G)\backslash\{uv\in E(G)|u\in N_G(V)\}$is called the graph obtained by deleting vertex v from G.



Vertex deletion at v

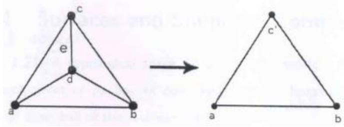
The subgraph G-e with $V(G\text{-}e)=V(G)$ and $E(G\text{-}e)=E(G)\backslash e$ is called the graph obtained by deleting edge e from G.



Edge deletion at e

The contraction of an edge e of the graph G, denoted by G/e is the graph obtained from G by the following steps:

1. delete vertices u and v from G.
2. insert a new vertex u′ such that $u'v\in E(G/e)$, for all $v\in N_G(u)\cup N_G(v)$.



Edge contraction {e}

Topological Wheel

A topological space M is called n-manifold if M is Hausdorff and each point of M has an open neighborhood homeomorphic to the n-dimensional open ball or the n-dimensional half-ball.

A triangulation of a topological space X is a homeomorphism h from the carrier of some simplical complex K to the space X. The image of a simplex of K under h is called a simplex of triangulation.

Any graph G can be represented by a topological space in the following sense:

1. V(G) is represented by a collection of distinct points in $\mathbb{R}^3$.
2. E(G) is represented by a collection of distinct, internally disjoint arcs, homeomorphic to the closed interval
   [0, 1] such that boundary points of the arcs represent the endpoints of the corresponding edge.

Let G be a graph (topological representation) and S_g, a surface of genus g. A graph embedding is a continuous one-to-one function $i : G\rightarrow S_g$ such that the function $i':G\rightarrow i(G)$ obtained by restricting the range of i is a homeomorphism.

Two graphs are said to be homemorphic if both can be obtained from the same graph by a sequence of subdivisions of edges.

A graph G is said to be planar if and only if it can be embedded on a sphere S.

A graph G is called connected if there exists a path from u to v, for all $u, v\in V(G)$.

If graph G is n-connected, $n\geq 2$, then every set of n points of G lie in a cycle.

By the definition of n-connected graph, G has no cutpoints and there must be a maximum of atleast n number of pairwise internally disjoint paths between any two vertices x, y in V(G). Thus, for any set $P=\{p_1, p_2,\ldots, p_n\}\subset V(G)$, we can find two internally disjoint paths between p_0 and p_n such that all $p_i(i\neq 1,n)$ lie in either of the two path. This gives a cycle containing P.

Kuratowski's theorem give the criterion for a graph to be planar. The graphs $K_{3,3}$ and K_5 forms the complete set of obstruction in planar embedding. These graphs are called the Kuratowski's graphs.

Theorem

(Kuratowski). A graph G is planar if and only if G has no subgraph homeomorphic to K_5 or $K_{3,3}$.

Proof

K_5 or $K_{3,3}$ are non-planar. So any graph containing a homeomorph of K_5or $K_{3,3}$ are non-planar. Thus the converse is proved.

Now we want to prove that, a graph G is non-planar then G has a subgraph homeomorphic to K_5 or $K_{3,3}$. Consider the (edge) minimal counter example H such that H is non-planar and does not contain a homeomorph of K_5 or $K_{3,3}$. The minimality ensures that removal of any edge makes H planar. We also assume that H does not have any vertex of valence 2, since a valence 2 vertex can be considered as the 1-subdivision of some edge and hence, we can smooth out valence 2 vertices to obtain H. First, we claim that H is at least 2-

connected or in other words, H does not have any cutpoints. To show this, suppose H has a cutpoint v, then removal of v disconnects H. Since H is non-planar, some component of H must be non-planar. This contradicts the minimality of H, and the claim follows.

Next, we claim that there exists edge e such that H-e has no cutpoints. To show this, assume that H has no such edge. Then, for any edge $e' \in E(H)$. H - e′ has a cutpoint. This means that H-e is 1-connected for all e, which in turn shows that H is 1-connected.This contradicts the first claim, and hence the claim holds.

Now, choose an edge e={u, v} of H whose removal does not affect connectivity. Consider H-e=H′ which is planar and 2-connected. Thus we can find a planar embedding of H′ with a cycle C such that C contains u and v and the number of regions enclosed by C is maximal among other embeddings. Let $C=v_0, v_1, \ldots, v_k=v, v_{k+1},\ldots, v_1, v_0$ and consider path P. From the maximality of C we can see that there is no path connecting two vertices in the set $\{v_0, v_1, \ldots, v_k\}$ that lies exterior to C and furthermore, there is no path connecting two vertices in the set $\{v_{k+1},\ldots, v_1, v_0\}$ that lies exterior to C.

The non-planarity of H implies that there is some structure inside cycle C that restricts the insertion of edge e between u and v.

A much more efficient planarity testing algorithm can be obtained from the following proof of Kuratowski's theorem due to Thomassen[6]. The idea of Thomassen's proof relies on the following result by Tutte which we state without proof [11].

Theorem

(Tutte). A graph G is 3-connected then it is a wheel or can be obtained from a wheel by a sequence of operations of the following two types.

1. The addition of new edge
2. The replacement of a vertex v having valence($\geq$ 4) by two adjacent points v′ and v′′ such that each point formerly joined to v is joined to exactly one of v′ and v′′ so that in the resulting graph, valence(v′) $\geq$ 3 and valence (v′′) $\geq$ 3).

The above proposition ensures the existence of wheel structure for the 3-connected graph in the following proof of Kuratowski'stheorem.

Theorem

Let G be a 3-connected graph with five or more vertices. Then there is some edge e of G such that the graph G/e is also 3-connected [10].

Proof

(Thomassen). Suppose for every edge e, the contracted graph G/e has a set of two vertices that disconnects it. One of those two vertices must be the vertex obtained by identifying the two endpoints of the edge e or else the same set of two vertices would also disconnect G, thereby contradicting the 3-connectivity of G. Thus for every edge e=uv together with some third vertex w disconnect G. Accordingly, let us choose an edge e and a vertex w such that the largest component H of the graph G – {u, v, w} is the largest for any disconnecting set consisting of three vertices, two of which are adjacent.

Let x be a vertex adjacent to w such that x lies in a component of G-{u, v, w} other than the maximum component H. Since vertices w and x are the endpoints of an edge of G, it follows that G has a disconnecting set of the form {w, x, y}. Now claim that some component of G – {w, x, y} is larger than H, a contradiction. To see this, let H′ be the subgraph of G induced by the vertices of H together with u and v. Since both u and v are adjacent to vertices of H (otherwise G would not be 3-connected), the subgraph H′ is connected. On one hand, perhaps the vertex y is not in H′. Since w and x are not in H′ either it follows that H′ is contained in a component of G –{w, x, y}, contradicting the maximality of H. On the other hand perhaps y is in H′. If H′ - y is connected, then there is again a contradiction of the maximality of H, since H′ - y has one more vertex than H.

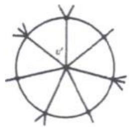
If H′ - y were not connected, then one component of H′ - y would contain both the vertices u and v, since u is adjacent to v and hence all the other components of H′ - y are connected to the rest of the graph G through the vertices y and w. This would imply that {y, w} disconnects G, contradicting the 3-connectivity of G. We conclude that for some edge e, the contracted graph G/e is 3-connected.

Corollary

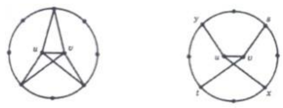
A graph G contains no homeograph of K_5 or $K_{3,3}$ then G is planar.

Proof

We prove by induction on the number of vertices. The statement is vacuously true of all graphs with four or fewer vertices. We assume that the statement is true for all graphs with fewer than n vertices, for $n \geq 5$.



Wheel structure with cycle C



Homeomorphs of K_5 and $K_{3,3}$

Consider the n+1 case, we can choose an edge e={u, v} such that G/e with the identified vertex v′ is still 3-connected. This means that G-v′ is 2-connected. Now consider the cycle C containing all the neighbours of v′. Now expand v′ back to u and v. By induction hypothesis, G will not contain the graphs. Since those graphs are homeomorphic to K_5 and $K_{3,3}$ as noted. Hence, G is planar.

CONCLUSION

Wheels, moving towards forward, upward, downward, connecting us in all space of life. The topological graph connected with wheels, makes us to move forward with new avenues.

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