Bayesian Shrinkage estimators of the multivariate normal distribution

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Abstract - This paper compares two shrinkage estimators of rates based on Bayesian methods. We estimate the mean θ of the multivariate normal distribution in $\mathfrak{R}^p$, when $\boldsymbol{\sigma^2}$ is unknown using the chi-square random variable. The Modified Bayes estimator $\boldsymbol{\delta_B^*}$and the Empirical Bayes estimator $\boldsymbol{\delta_{EB}^*}$ are considered and the limits of their risk ratios of the maximum likelihood estimator when n and p tend to infinity are obtained.

Keywords - *Bayes estimator, Empirical Bayes estimator, James-Stein estimator , Multivariate Gaussian random variable, Modified Bayes estimator, Shrinkage estimator..*

I. INTRODUCTION

Shrinkage estimation is a method to improve a raw estimator in some sense, by combining it with other information. Although the shrinkage estimator is biased, it is well known that it has minimum quadratic risk compared to natural estimators (mostly the maximum likelihood estimator) (Karamikabir, Afshari and Arashi, 2018). The shrinkage estimator, have evolved over time since their introduction by Stein in 1956, James and Stein in 1961 and Stein in 1981. In these works one estimates the mean θ by shrinking the empirical estimators of the mean, which are better in quadratic loss than the empirical mean estimator. More precisely, if X represents an observation of a sample drawn from a multivariate normal distribution, the aim is to estimate θ by an estimator δ relatively at the quadratic loss function $L(\delta,\theta) = \| \delta - \theta \|_p^2$ where $\|.\|_p$ the usual norm is in $\mathfrak{R}^p$ and the associated risk function given by: $\mathrm{R}(\delta,\theta) = E_\theta\big(L(\delta,\theta)\big)$.

James and Stein (1961), introduced a class of estimators by improving $\delta_\theta = X$, when the dimension of the space of the observations $p \geq 3$, is denoted by :

$$\delta_j^{JS} = \left(1 - \frac{p-2}{n+2}\frac{S^2}{\| X \|^2}\right)X_j, \qquad j = 1,2,\dots,p$$

where $S^2 \sim \sigma^2\chi_n^2$ is the estimate of σ^2. Baranchik (1964), proposed the positive-part of James-Stein estimator dominating the James-Stein estimator when $p \geq 3$,

$$\delta_j^{JS+} = max\left(\left(1 - \frac{p-2}{n+2}\frac{S^2}{\|X\|^2}\right)\right)X_j.$$

Casella and Hwang (1982), studied the case where σ^2 is known $\sigma^2 = 1$ and showed that if the limit of the ratio $\frac{\|\theta\|^2}{p}$, when p tends to infinity is a constant $c > 0$, then

$$\lim_{p\to+\infty}\frac{R(\delta_{JS},\theta)}{R(X,\theta)} = \lim_{p\to+\infty}\frac{R(\delta_{JS}{}^{+},\theta)}{R(X,\theta)} = \frac{c}{1+c}.$$

Sun (1995), has considered the following model: $(y_{ij}/\theta\ \sigma^2) \sim N(\theta_p,\sigma^2)$; $i = 1,2,\dots,n$, $j = 1,2,\dots,m$ where $E(y_{ij}) = \theta$ for the group j and $var(y_{ij}) = \sigma^2$ is unknown. The James-Stein estimator is written in $\delta^{JS} = (\delta_1^{JS},\dots,\delta_m^{JS})'$, where $\delta_j^{JS} = \left(1 - \frac{(m-3)}{(N+2)}\frac{S^2}{T^2}\right)(\bar{y}_j - \bar{y}) + \bar{y}$, $j = 1,2,\dots,m$, $S^2 = \sum_{i=1}^n\sum_{j=1}^m(y_{ij} - \bar{y}_j)^2$, $T^2 = n\sum_{i=0}^n(\bar{y}_j - \bar{y})^2$, $\bar{y}_i = \frac{\sum_{i=1}^n y_{ij}}{n}$ and $y = \frac{\sum_{i=1}^m \bar{y}_i}{m}$, $N = (n-1)m$.

He showed that for any estimator of the form $\delta^{JS} = (\delta_1^{JS},\dots,\delta_m^{JS})'$ where

$$\delta_j^{\psi} = \left(1 - \Psi(S^2, T^2)\right)\left(\bar{y}_j - \bar{y}\right) + \bar{y}, j = 1,2, \dots, m,$$

if $\lim_{p\to+\infty} \frac{\sum_{j=1}^{n}(\theta_j - \bar{\theta})^2}{m} = q$ exists, then $\lim_{m\to+\infty} \frac{R(\delta^{\psi},\theta)}{R(X,\theta)} \geq \frac{q}{q+\frac{\sigma^2}{n}}$ and

$$\lim_{m\to+\infty} \frac{R(\delta^{JS},\theta)}{R(X,\theta)} = \lim_{m\to+\infty} \frac{R(\delta^{JS+},\theta)}{R(X,\theta)} \geq \frac{q}{q+\frac{\sigma^2}{n}}.$$

where $\frac{q}{q+\frac{\sigma^2}{n}}$ constitutes a lower bound for the ratio $\lim_{m\to+\infty} \frac{R(\delta^{\psi},\theta)}{R(\delta_0,\theta)}$ and is equal to $\lim_{m\to+\infty} \frac{R(\delta^{JS},\theta)}{R(\delta_0,\theta)}$.

Sun (1995), also showed that this bound is attained for a class of estimators defined by: $\delta_j = \left(1 - \Psi(S^2, T^2)\right)\left(\bar{y}_j - \bar{y}\right) + \bar{y}, j = 1,2, \dots, m$ where Ψ satisfies certain conditions. This bound is also attained for any estimator dominating the James-Stein estimator, in particular the positive-part.

Further, we note that if n tends to infinity then the ratio $\frac{q}{q+\frac{\sigma^2}{n}}$ tends to 1, and thus the risk of the James-Stein estimator is that of δ_0 (when n and m tend to infinity).

Hamdaoui and Benmansour (2015), considered the following class of shrinkage estimators $\delta_{\psi} = \delta_{JS} + \iota\psi(S^2, \| X^2 \|)X$, which is introduced in Benmansour and Mourid (2007). The authors showed that if $\lim_{m\to+\infty} \frac{\|\theta\|^2}{p} = c(> 0)$ then the risk ratios $\frac{R(\delta_{\psi},\theta)}{R(X,\theta)}$, $\frac{R(\delta^{JS},\theta)}{R(X,\theta)}$ and $\frac{R(\delta^{JS+},\theta)}{R(X,\theta)}$ attain the lower bound $B_m = \frac{c}{1+c}$ when n and p tend to infinity provided that $\lim_{m\to+\infty} \frac{\|\theta\|^2}{p}$. Hamdaoui and Mezouar (2017), considered the general class of shrinkage estimators $\delta_{\psi} = \left(1 - \psi(S^2, \| X^2 \|)\frac{S^2}{\|X^2\|}\right)$ X. The authors showed the same results given by Hamdaoui and Benmansour (2015), under different conditions from the one given by Hamdaoui and Benmansour (2015). When the dimension p is finite, Brandwein and Strawderman (2012) considered the following model $(X, U) \sim f(\| X - \theta \|^2 + \| U \|^2)$, where $\dim X = \dim \theta = p$ and $\dim U = k$. The classical example of this model is, of course, the normal model of density $\left(\frac{1}{\sqrt{2\pi\sigma}}\right)^{p+k} e^{-\frac{|x-\theta|^2}{2\sigma^2}}$. They showed that the estimator $\delta = X + \left\{\frac{|U|^2}{k+2}\right\} g(X)$ dominates X, so that δ is Minimax, provided the function g satisfies certain conditions.

Maruyama (2014) has also studied the minimaxity of shrinkage estimator when the dimension of parameter's space is moderate. Then he considered the following model $Z \sim N(\theta, I_d)$ and the so called $\bar{\pi}$ -norm given by: $\| Z \|_p = \{\sum_{i=0}^{n} |z_i|^p\}^{\frac{1}{p}}$, $p > 0$. He studied the minimaxity of shrinkage estimators defined as follows: $\hat{\theta}_{\phi} = \left(\hat{\theta}_{1\phi}, \dots, \hat{\theta}_{d\phi}\right)$ with: $\hat{\theta}_{i\phi} = \left(1 - \phi\left(\| Z \|_p\right)/\phi\left(\| Z \|_p^{2-\alpha} |Z_i|^{\alpha}\right)\right)$ where $0 \leq \alpha \leq (d-2)/(d-1)$ and $p > 0$. Note that the risk functions of these estimators are calculated relatively to the usual quadratic loss function defined at above.

In this work we adopt the model $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ such that the parameter σ^2 is unknown and estimated by the statistic S^2: $S^2 \sim \sigma^2 \chi_n^2$ independent of the observations X. We give the prior distribution $\theta \sim N_p\left(\nu, \tau^2 I_p\right)$ where the hyperparameter υ is known and

the hyperparameter τ^2 is known or unknown. The aim is estimating the mean θ by a the Modified Bayes estimator δ_B^* when the hyperparameter τ^2 is known and by an Empirical Bayes estimator δ_{EB}^* when the hyperparameter τ^2 is unknown. Note that $R(X;\theta) = p\sigma^2$, is the risk of the Maximum likelihood estimator.

In section 1, we recall some technical's Lemmas that we use them for later and we also recall some results linked with Bayes estimators of the mean of a multidimensional normal distribution.

In Section 2, we give the main results of this paper. First, we take the prior low of θ: $\theta \sim N_p(\nu, \tau^2 I_p)$ where the hyperparameters ν, τ^2 are known and we construct a Modified Bayes estimator δ_B^* of the mean θ, then we study the minimaxity of this estimator when n and p are fixed. In the second part of this section we study the behaviour of the risk ratio of this estimator to the Maximum likelihood estimator X when n and p tend to infinity.

In the section 3, we take the prior distribution of θ: $\theta \sim N_p(\nu, \tau^2 I_p)$ where the hyperparameter ν is known and the hyperparameter τ^2 is unknown and we construct an Empirical Bayes estimators of the mean θ, then we will follow the same steps as we have given in section two. In the end we illustrate graphically the risk ratios of the Modified Bayes estimator δ_B^* to the Maximum likelihood estimator X for divers values of n.

II. PRELIMINARIES

We recall that if X is a multivariate Gaussian random $N_p(\theta, \sigma^2 I_p)$ in $\Re^p$ then $\frac{\|X\|^2}{\sigma^2} \sim \chi_q^2(\lambda)$ where $\chi_q^2(\lambda)$ denotes the non-central chi-square distribution with p degrees of freedom and non-centrality parameter $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$. We recall the following Lemmas that we will use often in our proofs.

Lemma 2.1 (Fourdrinier et. al., 2007)

Let $X \sim N_p(\theta, \sigma^2 I_p)$, $\theta \in \Re^p$. Then, for $p \geq 3$, we have $E\left(\frac{1}{\|X^2\|}\right) = \frac{1}{\sigma^2} E\left(\frac{1}{p-2+2k}\right)$, where $X \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$, being the Poisson's distribution of the parameter $\frac{\|\theta\|^2}{2\sigma^2}$.

Lemma 2.2 (Stein 1981)

Let Y be a $N(0,1)$ real random variable and let $g: \Re \to \Re$ be an indefinite integral of the Lesbegue measurable function g', essentially the derivative of g. Suppose also that $E(g'(Y)) < \infty$. Then $E(Yg(Y)) = E(g'(Y))$.

Lemma 2.3 (Casella and Hwang, 1982)

For any real function h such that $E\left[h\left(\chi_q^2(\lambda)\right)\chi_q^2(\lambda)\right]$ exists, we have

$$E\left\{h\left(\chi_q^2(\lambda)\right)\chi_q^2(\lambda)\right\} = qE\left\{h\left(\chi_{q+2}^2(\lambda)\right)\right\} + 2\lambda E\left\{h\left(\chi_{q+2}^2(\lambda)\right)\right\}$$

Lemma 2.4 (Benmansour and Hamdaoui, 2011)

Let f is a real function. If for $p \geq 3$, $E_{\chi_q^2(\lambda)}[f(U)]$ exists, then

a) If f is monotone non-increasing we have

$$E_{\chi_{p+2}^2(\lambda)}[f(U)] \leq E_{\chi_p^2(\lambda)}[f(U)] \tag{2.1}$$

b) If f is monotone non-decreasing we have

$$E_{\chi^2_{p+2}(\lambda)}[f(U)] \geq E_{\chi^2_p(\lambda)}[f(U)]. \tag{2.2}$$

We also recall in what follows a few results of Bayes estimator.

Let $X/\theta \sim N_p(\theta, \sigma^2 I_p)$ and $\theta \sim N_p(\nu, \tau^2 I_p)$ where σ^2 known, and hyperparameters is ν, τ^2 are known. Then from Lindley (1972), we have

$\theta/X \sim N_p(\upsilon + B(X-\upsilon), \sigma^2 B I_p)$ where $B = \frac{\tau^2}{\tau^2+\sigma^2}$, then the Bayes estimator of θ is

$\delta_B(X) = E\left(\frac{\theta}{X}\right) = \upsilon + B(X-\upsilon)$ thus

$$\delta_B(X) = \left(1 - \frac{\tau^2}{\tau^2+\sigma^2}\right)(X-\upsilon) +. \tag{2.3}$$

We deduce that

i) $R(\delta_B(X);\theta) = (1-B)^2 \parallel \theta - \upsilon \parallel^2 + B^2 p\sigma^2$
and

ii) $\frac{R(\delta_B(X);\nu,\tau^2,\sigma^2)}{R(X)} = \frac{\tau^2}{\tau^2+\sigma^2}$

III. RESULTS AND DISCUSSION

In this section we are interested in studying the minimaxity, bounds and limits of risk ratios of the Modified Bayes and the Empirical Bayes estimators, to the Maximum likelihood estimator X. In the next, we give the following Lemma.

Lemma 3.1

For any $c > 0$, we have:

$$\frac{1}{n+2+c} \leq E_{\chi^2_{n+2}}\left(\frac{1}{u+c}\right) \leq \frac{1}{n+c}, \tag{3.1}$$

$$\frac{1}{(n+4+c)^2} \leq E_{\chi^2_{n+4}}\left[\frac{1}{(u+c)^2}\right] \leq \frac{1}{(n+c)^2} \tag{3.2}$$

Proof

From Jensen's inequality we have

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$$E_{\chi^2_{n+2}}\left(\frac{1}{u+c}\right) \geq \frac{1}{n+2+c}.$$

On the other hand, from lemma 2.3, we have

$$1 = E_{\chi^2_n}\left(\frac{u}{u+c}\right) + cE_{\chi^2_{n+2}}\left(\frac{1}{u+c}\right)$$
$$= nE_{\chi^2_{n+2}}\left(\frac{1}{u+c}\right) + cE_{\chi^2_n}\left(\frac{1}{u+c}\right)$$

Then

$$E_{\chi^2_{n+2}}\left(\frac{1}{u+c}\right) = \frac{1}{n}\left[1 - cE_{\chi^2_n}\left(\frac{1}{u+c}\right)\right] \leq \frac{1}{n+c}.$$

The last inequality follows from Jensen's inequality. The proof of the formula (3.2) is as follows: from Jensen's inequality we have

$$E_{\chi^2_{n+4}}\left[\frac{1}{(u+c)^2}\right] \leq \frac{1}{(n+4+c)^2}.$$

In other hand, we have

$$E_{\chi^2_{n+2}}\left(\frac{1}{u+c}\right) = E_{\chi^2_{n+2}}\left(\frac{u}{(u+c)^2}\right) + cE_{\chi^2_{n+2}}\left(\frac{1}{(u+c)^2}\right)$$
$$= (n+2)E_{\chi^2_{n+4}}\left(\frac{1}{(u+c)^2}\right) + cE_{\chi^2_{n+2}}\left(\frac{1}{(u+c)^2}\right) \qquad (3.3)$$
$$\geq (n+2)E_{\chi^2_{n+4}}\left(\frac{1}{(u+c)^2}\right) + cE_{\chi^2_{n+2}}\left(\frac{1}{(u+c)^2}\right) \qquad (3.4)$$
$$\geq (n+2+C)E_{\chi^2_{n+4}}\left(\frac{1}{(u+c)^2}\right)$$

The equality (3.3) follows from Lemma 2.3 and the inequality (3.4) follows to the formula (2.1) of Lemma 2.4. Hence

$$E_{\chi^2_{n+2}}\left(\frac{1}{(u+c)^2}\right) \leq \frac{1}{n+2+c}E_{\chi^2_{n+2}}\left(\frac{1}{u+c}\right)$$

Using the formula (3.1), we obtain

$$E_{\chi^2_{n+2}}\left(\frac{1}{(u+c)^2}\right) \leq \frac{1}{n+2+c}\frac{1}{n+c} \leq \frac{1}{(n+c)^2}.$$

Modified Bayes Estimator

Let $X/\theta \sim N_p(\theta, \sigma^2 I_p)$ and $\theta \sim N_p(\nu, \tau^2 I_p)$ where σ^2 is unknown, and the hyperparameters ν, τ^2 are known. We note that $\frac{S^2}{S^2+n\tau^2}$ is an asymptotically unbiased estimator of ratio $\frac{\sigma^2}{\sigma^2+\tau^2}$. Endeed

$$E\left(\frac{S^2}{S^2+n\tau^2}\right) = E\left(\frac{\frac{S^2}{\sigma^2}}{\frac{S^2}{\sigma^2}+n\frac{\tau^2}{\sigma^2}}\right) = nE_{\chi^2_{n+2}}\left(\frac{S^2}{u+n\frac{\tau^2}{\sigma^2}}\right)$$

The last equality comes from Lemma 2.3. From the previous equality and the formula (3.1) of Lemma 3.1, we obtain

$$\frac{n}{n\left(1+\frac{\tau^2}{\sigma^2}\right)+2} \leq E\left(\frac{S^2}{S^2+n\tau^2}\right) \leq \frac{n}{n\left(1+\frac{\tau^2}{\sigma^2}\right)},$$

It is clear that both the upper and the lower bound converge to $\frac{\sigma^2}{\sigma^2+\tau^2}$ when $n \to \infty$. Thus

$$\lim_{n\to\infty} E\left(\frac{S^2}{S^2+n\tau^2}\right) = \frac{\sigma^2}{\sigma^2+\tau^2}.$$

If we replace in formula (2.3) the ratio $\frac{\sigma^2}{\sigma^2+\tau^2}$ by its estimator $\frac{S^2}{S^2+n\tau^2}$, we obtain the Modified Bayes estimator expressed as

$$\delta_B^* = \left(1 - \frac{S^2}{S^2+n\tau^2}\right)(X-\upsilon) + \upsilon \qquad (3.5)$$

Proposition 3.2

Let the Modified Bayes estimator δ_B^* given in (3.5), then

i) The risk function of the estimator δ_B^* is

$$R(\delta_B^*; \nu, \tau^2, \sigma^2) = p\sigma^2 + p\sigma^2\left\{n(n+2)\left(1+\frac{\tau^2}{\sigma^2}\right)E_{\chi^2_{n+4}}\left(\frac{1}{\left(u+n\frac{\tau^2}{\sigma^2}\right)^2}\right) - 2nE_{\chi^2_{n+2}}\left(\frac{1}{u+n\frac{\tau^2}{\sigma^2}}\right)\right\}$$

ii) $1 + \frac{n(n+2)\left(1+\frac{\tau^2}{\sigma^2}\right)}{\left(n\left(1+\frac{\tau^2}{\sigma^2}\right)+4\right)^2} - \frac{2}{1+\frac{\tau^2}{\sigma^2}} \leq \frac{R(\delta_B^*;\nu,\tau^2,\sigma^2)}{R(X)} \leq 1 + \frac{(n+2)}{n\left(1+\frac{\tau^2}{\sigma^2}\right)} - \frac{2n}{n\left(1+\frac{\tau^2}{\sigma^2}\right)+2}.$

Proof

We have

$$R(\delta_B^*;\theta)=E_\theta\left(\left\|\left(1-\frac{S^2}{S^2+n\tau^2}\right)(X-\upsilon)+\upsilon-\theta\right\|^2\right).$$

From the independence to two variables X and S^2, we have

$$R(\delta_B^*;\theta)=E_\theta(\|X-\theta\|^2)+E_\theta\left(\frac{S^2}{S^2+n\tau^2}\right)^2E_\theta(\|X-\upsilon\|^2)-2\sigma^2E_\theta\left(\frac{S^2}{S^2+n\tau^2}\right)E_\theta\left(\langle\frac{X-\theta}{\sigma},\frac{X-\nu}{\sigma}\rangle\right)$$

$$=p\sigma^2+E_{\chi^2_{n+2}}\left(\frac{u}{u+n\frac{\tau^2}{\sigma^2}}\right)^2\left\{p\sigma^2+\|\theta-\upsilon\|^2-2np\sigma^2E_{\chi^2_n}\left(\frac{u}{u+n\frac{\tau^2}{\sigma^2}}\right)\right\} \tag{3.6}$$

$$=p\sigma^2+n(n+2)E_{\chi^2_{n+4}}\left(\frac{1}{\left(u+n\frac{\tau^2}{\sigma^2}\right)^2}\right)(p\sigma^2+\|X-\theta\|^2-2np\sigma^2)E_{\chi^2_{n+2}}\left(\frac{1}{u+n\frac{\tau^2}{\sigma^2}}\right).$$

The equality (3.6) according to the Lemma 2.2 and the last equality follow from Lemma 2.3. Thus

$$R(\delta_B^*;\nu,\tau^2,\sigma^2)=E_{\delta_B^*;\nu,\tau^2,\sigma^2}[R(\delta_B^*;\theta)]$$

$$=p\sigma^2+n(n+2)E_{\chi^2_{n+4}}\left(\frac{1}{\left(u+n\frac{\tau^2}{\sigma^2}\right)^2}\right)\left(p\sigma^2+E_{\chi^2_{\upsilon,\tau^2}}(|X-\theta|^2)-2np\sigma^2\right)E_{\chi^2_{n+2}}\left(\frac{1}{u+n\frac{\tau^2}{\sigma^2}}\right)$$

$$=p\sigma^2+p\sigma^2\left\{n(n+2)\left(1+\frac{\tau^2}{\sigma^2}\right)E_{\chi^2_{n+4}}\left(\frac{1}{\left(u+n\frac{\tau^2}{\sigma^2}\right)^2}\right)-2nE_{\chi^2_{n+2}}\left(\frac{1}{u+n\frac{\tau^2}{\sigma^2}}\right)\right\}$$

From i) we have

$$\frac{R(\delta_B^*;\nu,\tau^2,\sigma^2)}{R(X)}=1+n(n+2)\left(1+\frac{\tau^2}{\sigma^2}\right)E_{\chi^2_{n+2}}-2nE_{\chi^2_{n+2}}\left(\frac{1}{u+n\frac{\tau^2}{\sigma^2}}\right)$$

Using formulas (3.1) and (3.2) of Lemma 3.1, we obtain

$$\frac{R(\delta_B^*;\nu,\tau^2,\sigma^2)}{R(X)}\le 1+n(n+2)\left(1+\frac{\tau^2}{\sigma^2}\right)\frac{1}{\left(n+n\frac{\tau^2}{\sigma^2}\right)^2}-\frac{2n}{n+2+n\frac{\tau^2}{\sigma^2}}$$

$$\le 1+\frac{(n+2)}{n\left(1+\frac{\tau^2}{\sigma^2}\right)}-\frac{2n}{n\left(1+\frac{\tau^2}{\sigma^2}\right)+2}$$

and

$$\frac{R(\delta_B^*;\nu,\tau^2,\sigma^2)}{R(X)}\ge 1+n(n+2)\left(1+\frac{\tau^2}{\sigma^2}\right)\frac{1}{\left(n+4+n\frac{\tau^2}{\sigma^2}\right)^2}-\frac{2n}{n+2+n\frac{\tau^2}{\sigma^2}}$$

$$\ge 1+\frac{n\left(n+21+\frac{\tau^2}{\sigma^2}\right)}{\left(n\left(1+\frac{\tau^2}{\sigma^2}\right)+4\right)^2}-\frac{2}{1+\frac{\tau^2}{\sigma^2}}$$

Theorem 3.3

a) *If* $p\ge 3$ and $n\ge 5$ the estimator δ_B^* given in (3.5) is minimax,

b) $\lim_{n,p\to\infty}\frac{R(\delta_B^*;\nu,\tau^2,\sigma^2)}{R(X)}=\frac{\tau^2}{\tau^2+\sigma^2}$.

Proof

a) From the previous Proposition, we have

$$R(\delta_B^*;\nu,\tau^2,\sigma^2)\le p\sigma^2+p\sigma^2\left\{\left(\frac{n(n+2)\left(1+\frac{\tau^2}{\sigma^2}\right)}{\left(n+n\frac{\tau^2}{\sigma^2}\right)^2}\right)-\left(\frac{2n}{n+2+n\frac{\tau^2}{\sigma^2}}\right)\right\}$$

The study of the variation of the real function $h(x) = \frac{x+2}{x\left(1+\frac{\tau^2}{\sigma^2}\right)} - \frac{2n}{x\left(1+\frac{\tau^2}{\sigma^2}\right)+2}$, shows that

$\frac{(n+2)}{n\left(1+\frac{\tau^2}{\sigma^2}\right)} - \frac{2n}{n\left(1+\frac{\tau^2}{\sigma^2}\right)+2} \leq 0$ for any $n \geq 5$,

Then $R(\delta_B^*; \nu, \tau^2, \sigma^2) \leq R(X) \leq p\sigma^2$ for any $n \geq 5$. Thus δ_B^* is minimax for any $n \geq 5$.

b) Immediately from ii) of the Proposition 3.2.

Empirical Bayes estimator

Let $X/\theta \sim N_p(\theta, \sigma^2 I_p)$ and $\theta \sim N_p(\nu, \tau^2 I_p)$ where σ^2 is unknown and the hyperparameter ν is known and the hyperparameter $\tau^2$is unknown. We note that $\frac{p-2}{n+2}\frac{S^2}{\|X-\upsilon\|^2}$ (where$S^2 \sim \sigma^2\chi_n^2$) is an asymptotically unbiased estimator of ratio$\frac{\sigma^2}{\sigma^2+\tau^2}$. Endeed: from the independence to two variables $S^2$and$\|X\|^2$, we have

$$E\left(\frac{p-2}{n+2}\frac{S^2}{\|X-\upsilon\|^2}\right) = \frac{p-2}{n+2}\frac{\sigma^2}{\sigma^2+\tau^2}E\left(\frac{S^2}{\sigma^2}\right)E\left(\frac{1}{\frac{\|X-\upsilon\|^2}{\sigma^2+\tau^2}}\right)$$

As the marginal distribution of X is $X \sim N_p(\upsilon, (\sigma^2+\tau^2)I_p)$, then $\frac{\|X-\upsilon\|^2}{\sigma^2+\tau^2} \sim \chi_p^2$.

Using the Lemma 2.1, we obtain $E\left(\frac{1}{\frac{\|X-\upsilon\|^2}{\sigma^2+\tau^2}}\right) = E\left(\frac{1}{\chi_p^2}\right) = \frac{1}{p-2}$, thus

$$E\left(\frac{p-2}{n+2}\frac{S^2}{\|X-\upsilon\|^2}\right) = \frac{n}{n+2}\frac{\sigma^2}{\sigma^2+\tau^2} \xrightarrow[n\to\infty]{} \frac{\sigma^2}{\sigma^2+\tau^2}.$$

If we replace in formula (2.3) the ratio$\frac{\sigma^2}{\sigma^2+\tau^2}$ by its estimator$\frac{p-2}{n+2}\frac{S^2}{\|X-\upsilon\|^2}$, we obtain the Empirical Bayes estimator expressed as

$$\delta_{EB}^* = \left(1 - \frac{p-2}{n+2}\frac{S^2}{\|X-\upsilon\|^2}\right)(X-\upsilon) + \upsilon \tag{3.7}$$

Proposition 3.4

The risk function of the Empirical Bayes estimator δ_{EB}^* given in (3.7) is

$$R(\delta_{EB}^*; \nu, \tau^2, \sigma^2) = p\sigma^2\left[1 - \frac{p-2}{p}\frac{n}{n+2}\frac{\sigma^2}{\sigma^2+\tau^2}\right].$$

Proof

$$R(\delta_{EB}^*; \nu, \tau^2, \sigma^2) = E\left(\left\|\left(1 - \frac{p-2}{n+2}\frac{S^2}{\|X-\upsilon\|^2}\right)(X-\upsilon) + \upsilon - \theta\right\|^2\right)$$

$$= E(\|X-\theta\|^2) + \left(\frac{p-2}{n+2}\right)^2\frac{\sigma^4}{\sigma^2+\tau^2}E\left(\frac{S^2}{\sigma^2}\right)E\left(\frac{1}{\frac{\|X-\upsilon\|^2}{\sigma^2+\tau^2}}\right)$$

$$-\frac{n(p-2)}{n+2}\sigma^2 E\left(\frac{S^2}{\sigma^2}\right)E\left[\frac{X-\theta}{\sigma}, \frac{1}{\frac{\|X-\upsilon\|^2}{\sigma^2}}\frac{X-\upsilon}{\sigma}\right]$$

Using Lemma 2.2, we obtain

$$E\left[\frac{X-\theta}{\sigma}, \frac{1}{\frac{\|X-\upsilon\|^2}{\sigma^2}}\frac{X-\upsilon}{\sigma}\right] = \sum_{i=1}^{p} E\left(\frac{X-\theta}{\sigma}\left(\frac{1}{\frac{\|X-\upsilon\|^2}{\sigma^2}}\frac{X-\upsilon}{\sigma}\right)\right) = \sum_{i=1}^{p} E\left[\frac{\partial}{\partial X_i}\left(\frac{1}{\frac{\|X-\upsilon\|^2}{\sigma^2}}\frac{X-\upsilon}{\sigma}\right)\right]$$

$$= (p-2)E\left(\frac{1}{\frac{\|X-\upsilon\|^2}{\sigma^2}}\right) = (p-2)\frac{\sigma^2}{\sigma^2+\tau^2}E\left(\frac{1}{\frac{\|X-\upsilon\|^2}{\sigma^2}}\right)$$

$$= (p-2)\frac{\sigma^2}{\sigma^2+\tau^2}E\left(\frac{1}{\chi_p^2}\right) = \frac{\sigma^2}{\sigma^2+\tau^2}$$

Because $\frac{\|X-\upsilon\|^2}{\sigma^2} \sim \chi_p^2$ and $E\left(\frac{1}{\chi_p^2}\right) = \frac{1}{p-2}$. Thus, $R(\delta_{EB}^*; \nu, \tau^2, \sigma^2) = p\sigma^2\left[1 - \frac{p-2}{p}\frac{n}{n+2}\frac{\sigma^2}{\sigma^2+\tau^2}\right]$.

Theorem 3.5

Let the Empirical Bayes estimator δ_{EB}^* given in (3.7), then

a) *If* $p \geq 3$, the estimator δ_{EB}^* is minimax,

b) $\lim_{n,p\to\infty} \frac{R(\delta_{EB}^*;\nu,\tau^2,\sigma^2)}{R(X)} = \frac{\tau^2}{\tau^2+\sigma^2}$

Proof

Immediately from the Proposition 3.4.

IV. SIMULATION RESULTS

We illustrate graphically the performance of the risk ratios of the Bayes estimator δ_B^* to the Maximum likelihood estimator x expressed as $\frac{R(\delta_B^*;\nu,\tau^2,\sigma^2)}{R(X)}$ as a function of $\lambda = \frac{\tau^2}{\sigma^2}$ for various values of n.

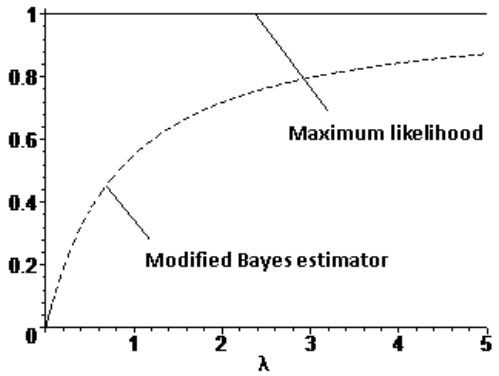

Fig. 1. Graph of risk ratio $\frac{R(\delta_B^*;\nu,\tau^2,\sigma^2)}{R(X)}$ as function of $\lambda = \frac{\tau^2}{\sigma^2}$ for $n = 5$.

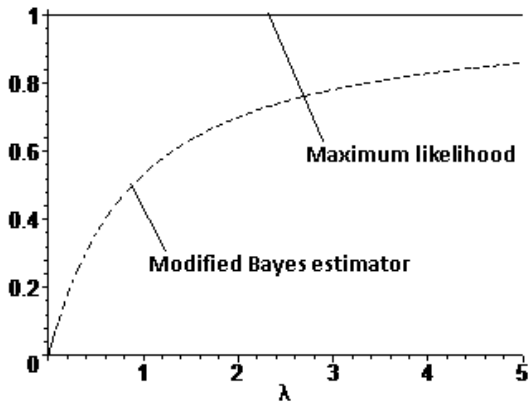

Fig. 2. Graph of risk ratio $\frac{R(\delta_B^*;\nu,\tau^2,\sigma^2)}{R(X)}$ as function of $\lambda = \frac{\tau^2}{\sigma^2}$ for $n = 8$.

V. CONCLUSIONS

In this paper, we study the asymptotic behaviour of the risk ratios of shrinkage estimators of the mean θ of a multivariate Gaussian random variable in $\Re^p$ under a quadratic loss

function. We take the same model $X \sim N_p(\theta, \sigma^2 I_p)$ with the unknown σ^2 estimated by the statistic $S^2 \sim \sigma^2 \chi_n^2$ independent of X, given the prior distribution $\theta \sim N_p(\nu, \tau^2 I_p)$ where the hyperparameter υ is known and the hyperparameter τ^2 is known or unknown, then we constructed the Modified Bayes estimator δ_B^* when the hyperparameter τ^2 is known and the Empirical Bayes estimator δ_{EB}^* when the hyperparameter τ^2 is unknown and showed that the estimators δ_B^* and δ_{EB}^* are Minimax when n and p are finite. When n and p tend simultaneously to infinity without assuming any order relation or functional relation between n and p, we showed that the risk ratios $\frac{R(\delta_B^*;\nu,\tau^2,\sigma^2)}{R(X)}$ and $\frac{R(\delta_{EB}^*;\nu,\tau^2,\sigma^2)}{R(X)}$ tend to the same value $\frac{\tau^2}{\tau^2+\sigma^2}$ which is less than 1. An idea would be to see whether one can obtain similar results of the asymptotic behaviour of risk ratios in the general case of the symmetrical spherical models, for general classes of shrinkage estimators.

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