An Integral Involving Generalized Multivariable Mittag-Leffler Function

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ABSTRACT: *The aim of the present paper is to establish new integrals involving generalized multivariable Mittag-Leffler function* $E_{\rho_j,\beta,p_j}^{\gamma_j,l_j,q_j}(z)$ *associated with a general class of polynomials. The integral obtained is unified in nature and act as key formula from which we can obtain as their special cases, integral formulae concerning a large number of simpler special functions and polynomials. For the sake of illustration, we record here three corollaries as special cases of our main formula which are also new and of interest by themselves. The findings of the present results are basic in nature and are likely to find useful applications in several fields*

I. INTRODUCTION

The Swedish mathematician Gosta Mittag- Leffler [3] in 1903, introduced the function $E_\rho(z)$, defined as

$$E_\rho(z)=\sum_{n=0}^{\infty}\frac{(z)^n}{\Gamma(\rho n+1)} \qquad \{\rho, z\in C; \mathrm{Re}(\rho)>0\} \tag{1.1}$$

where Γ(z) is the familiar Gamma function.The Mittag- Leffler function (1.1) reduces immediately to the exponential function $e^z = E_1(z)$ when α=1. Mittag- Leffler function naturally occurs as the solution of fractional order differential equation or fractional order integral equations.

During the last century and due to its involvement in the problems of physics,engineering, and applied sciences, many authors defined and studied in their research papers different generalizations of Mittag-leffler type function,namely $E_{\rho,\beta}(z)$ as Wiman[10] function, $E_{\rho,\beta}^{\gamma}(z)$ stated by Prabhakar [4] , $E_{\rho,\beta}^{\gamma,p}(z)$ defined and studied by Shukla and Prajapati [6] ,and $E_{\rho,\beta,q}^{\gamma,l,p}(z)$ investigated by Salim and Faraj[5].

The multivariate analogue of generalized Mittag-Leffler function $E_{\rho,\beta}^{\gamma}(z)$ is setup and studied by Saxena et al. [1] in the following form

$$E_{\rho_j,\beta}^{\gamma_j}(z_1,\dots\dots z_m)=E_{(\rho_1,\dots,\rho_m),\beta}^{(\gamma_1,\dots,\gamma_m)}(z_1,\dots\dots z_m)$$
$$=\sum_{r_1,\dots,r_m=0}^{\infty}\frac{(\gamma_1)_{r_1}\dots(\gamma_m)_{r_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_j r_j)}\frac{(z_1)^{r_1}\dots(z_m)^{r_m}}{r_1!\dots r_m!} \tag{1.2}$$

where

$\beta,\rho_j,\gamma_j\in C$ *and* $\mathrm{Re}(\rho_j)>0;\ j=0,1,2,\dots m.$

A further generalization of multivariate analogue of generalized Mittag-Leffler function $E_{\rho,\beta,q}^{\gamma,l,p}(z)$ was also mentioned,Saxena et al.[1]in terms of the following multiple series:

$$E_{(\rho_1,\ldots,\rho_m),\beta}^{(\gamma_1,\ldots,\gamma_m;l_1,\ldots,l_m)}(z_1,\ldots,z_m)=\sum_{r_1,\ldots,r_m=0}^{\infty}\frac{(\gamma_1)_{l_1r_1}\ldots(\gamma_m)_{l_mr_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_jr_j)}\frac{(z_1)^{r_1}\ldots(z_m)^{r_m}}{r_1!\ldots r_m!} \tag{1.3}$$

where

$\beta,\rho_j,\gamma_j,l_j\in C\ and\ (\{\mathrm{Re}(\rho_j),\mathrm{Re}(l_j)\})>0;\ j=0,1,2,\ldots m.$

Recently Meena et al [2] introduced a multivariable generalization of Mittag- Leffler function as

$$E_{(\rho_j;\beta;q_j)}^{(\gamma_j;l_j;p_j)}(z_1,\ldots z_m)=\sum_{r_1,\ldots,r_m=0}^{\infty}\frac{(\gamma_1)_{p_1r_1}\ldots(\gamma_m)_{p_mr_m}}{\Gamma(\beta+\sum_{j=1}^{m}\rho_jr_j)}\frac{(z_1)^{r_1}\ldots(z_m)^{r_m}}{(l_1)_{q_1r_1}\ldots\ldots(l_m)_{q_mr_m}} \tag{1.4}$$

where

$$\beta,\rho_j,\gamma_j,l_j\in C;\ \min_{1\le j\le m}\{\mathrm{Re}(\beta),\mathrm{Re}(\rho_j),\mathrm{Re}(\gamma_j),\mathrm{Re}(l_j)\}>0$$
$$and\ p_j,q_j>0;\ p_j<q_j+\mathrm{Re}(\rho_j);\ j=0,1,2,\ldots m. \tag{1.5}$$

Equation (1.4) is just a multivariable generalized formula of Mittag-Leffler function ; its various properties including differentiation,Laplace ,Beta, and Mellin transforms,and generalized hypergeometric series form and its relationship with other type of special functions were investigated and established by Meena etal.

General class of polynomial $S_V^U[x]$ introduced by Srivastava [7] is defined as:

$$S_V^U[x]=\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}A_{V,R}\,x^R}{\Gamma(R+1)} \tag{1.6}$$

Where U is an arbitrary positive integer,the cofficient $A_{V,R}$ are arbitrary constants,real or complex and $(a)_n$ denotes the Pochhammer symbol defined by

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}1 & if\ n=0\\ a(a+1)\ldots\ldots(a+n-1) & \forall n\in\{1,2,3\ldots\}\end{cases}$$

If x=0 $A_{0,0}=1$,then $S_V^U[x]$ reduces to unity.

If we take U=2, $A_{V,R}=(-1)^R$ in (1.6) the General class of polynomial is reduced to Hermite Polynomial [9, p.106, Eq(5.54)].

$$S_V^U[x] \to x^{\frac{V}{2}} H_V\left(\frac{1}{2\sqrt{x}}\right) \tag{1.7}$$

$$H_V(x) = \sum_{R=0}^{\left[\frac{V}{2}\right]} \frac{(-1)^R (2x)^{V-2R} \Gamma(V+1)}{\Gamma(R+1)\Gamma(V-2R+1)} \tag{1.8}$$

If we take U=1, $A_{V,R} = \frac{(V+1)_R}{\Gamma(R+1)\Gamma(R+1)}$ in (1.6) the General class of polynomial is reduced to Bateman Polynomials [8, p.183, Eq.(42);1,pp.574 &575]

$$S_V^U[x] \to Z_V(x) \tag{1.9}$$

$$Z_V(x) = {}_2F_2\left(\begin{matrix} -V, V+1; \\ 1\ ,1\ \ ; \end{matrix} x\right) \tag{1.10}$$

If we take U=1, $A_{V,R} = \binom{V+\alpha}{V} \frac{1}{(\alpha+1)_R}$ in (1.6) the General class of polynomial is reduced to Laguerre Polynomials [9, p.101, Eq.(5.1.6)]

$$S_V^1[x] \to L_V^{(\alpha)}(x) \tag{1.11}$$

$$L_V^{(\alpha)}(x) = \sum_{R=0}^{V} \binom{V+\alpha}{V} \frac{(-V)_R}{(\alpha+1)_R} \frac{(x)^R}{\Gamma(R+1)} \tag{1.12}$$

This paper is devoted for the study new integrals of generalized multivariable Mittag-Leffler function $E_{\rho_j,\beta,q_j}^{\gamma_j,l_j,p_j}(z)$ defined in (1.4) associated with a general class of polynomials.

II. RESULT REQUIRED

In this section, we establish an integral formula involving the multivariable generalized Mittag-Leffler function $E_{\rho_j,\beta,q_j}^{\gamma_j,l_j,p_j}(z)$ defined in (1.4), which is required to establish main result

Theorem 1

If $\rho_j, \gamma_j, l_j, \lambda, \delta \in C;\ \min\{\mathrm{Re}(\rho_j), \mathrm{Re}(\gamma_j), \mathrm{Re}(l_j), \mathrm{Re}(\delta)\} > 0$ *and* $p_j, q_j > 0; l_j \in N, (j = 0,1,\ldots m)$ *and* $\lambda \notin Z_0^-$, *then*

$$\frac{1}{\Gamma\delta}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1} E_{\rho_j,\lambda,q_j}^{\gamma_j,l_j,p_j}\left[z_1 u^{\rho_1},\ldots\ldots z_m u^{\rho_m}\right]du = E_{\rho_j,\lambda+\delta,q_j}^{\gamma_j,l_j,p_j}\left[z_1,\ldots\ldots z_m\right] \tag{2.1}$$

Proof:

$$\frac{1}{\Gamma\delta}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1} E_{\rho_j,\lambda,q_j}^{\gamma_j,l_j,p_j}\left[z_1u^{\rho_1},......z_mu^{\rho_m}\right]du=$$

$$=\frac{1}{\Gamma(\delta)}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}\sum_{r_1,....,r_m=0}^{\infty}\frac{(\gamma_1)_{p_1r_1}.....(\gamma_m)_{p_mr_m}}{\Gamma(\lambda+\sum_{j=1}^{m}\rho_jr_j)}\frac{(z_1)^{r_1}....(z_m)^{r_m}(u)^{\sum_{j=1}^{m}\rho_jr_j}}{(l_1)_{q_1r_1}.........(l_m)_{q_mr_m}}du$$

$$=\frac{1}{\Gamma(\delta)}\sum_{r_1,....,r_m=0}^{\infty}\frac{(\gamma_1)_{p_1r_1}.....(\gamma_m)_{p_mr_m}}{\Gamma(\lambda+\sum_{j=1}^{m}\rho_jr_j)}\frac{(z_1)^{r_1}....(z_m)^{r_m}}{(l_1)_{q_1r_1}.........(l_m)_{q_mr_m}}\int_0^1 u^{\lambda+\sum_{j=1}^{m}\rho_jr_j-1}(1-u)^{\delta-1}du$$

$$=\frac{1}{\Gamma(\delta)}\sum_{r_1,....,r_m=0}^{\infty}\frac{(\gamma_1)_{p_1r_1}.....(\gamma_m)_{p_mr_m}}{\Gamma(\lambda+\sum_{j=1}^{m}\rho_jr_j)}\frac{(z_1)^{r_1}....(z_m)^{r_m}}{(l_1)_{q_1r_1}.........(l_m)_{q_mr_m}}B(\delta,\lambda+\sum_{j=1}^{m}\rho_jr_j)$$

$$= E_{\rho_j,\lambda+\delta,q_j}^{\gamma_j,l_j,p_j}\left[z_1,......z_m\right] \tag{2.2}$$

This complete the proof of Theorem 1

III. MAIN RESULT

Theorem2

If $\rho_j,\gamma_j,l_j,\lambda,\delta\in C;\ \min\{\mathrm{Re}(\rho_j),\mathrm{Re}(\gamma_j),\mathrm{Re}(l_j),\mathrm{Re}(\delta)\}>0$ *and* $p_j,q_j>0;l_j\in N,\ (j=0,1,...m)$ *and* $\lambda\notin Z_0^-$, *then*

$$\frac{1}{\Gamma\delta}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}S_{V_1}^{U_1}\left[\sigma_1(1-u)\right]S_{V2}^{U2}\left[\sigma_2(1-u)\right]E_{\rho_j,\lambda,q_j}^{\gamma_j,l_j,p_j}\left[z_1u^{\rho_1},......z_mu^{\rho_m}\right]du$$
$$=\frac{\Gamma(\delta+R_1+R_2)}{\Gamma(\delta)}\Delta_{U_2,V_2,\sigma_2,R_2}^{U_1,V_1,\sigma_1,R_1}E_{\rho_j,\lambda+\delta+R_1+R_2,q_j}^{\gamma_j,l_j,p_j}\left[z_1,......z_m\right] \tag{3.1}$$

where

$$\Delta_{U_2,V_2,\sigma_2,R_2}^{U_1,V_1,\sigma_1,R_1}=\sum_{R_1=0}^{\left[\frac{V_1}{U_1}\right]}\frac{(-V_1)_{U_1R_1}A_{V_1,R_1}\sigma_1^{R_1}}{\Gamma(R_1+1)}\sum_{R_2=0}^{\left[\frac{V_2}{U2}\right]}\frac{(-V_2)_{U_2R_2}A_{V_2,R_2}\sigma_2^{R_2}}{\Gamma(R_2+1)} \tag{3.2}$$

Proof:

$$\frac{1}{\Gamma\delta}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}\, S_{V_1}^{U_1}\left[\sigma_1(1-u)\right] S_{V2}^{U2}\left[\sigma_2(1-u)\right] E_{\rho_j,\lambda,q_j}^{\gamma_j,l_j,p_j}\left[z_1u^{\rho_1},......z_m u^{\rho_m}\right]du$$

$$=\frac{1}{\Gamma\delta}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}\sum_{R_1=0}^{\left[\frac{V_1}{U_1}\right]}\frac{(-V_1)_{U_1R_1}\,A_{V_1,R_1}\,\sigma_1^{R_1}(1-u)^{R_1}}{\Gamma(R_1+1)}\sum_{R_2=0}^{\left[\frac{V_2}{U2}\right]}\frac{(-V_2)_{U_2R_2}\,A_{V_2,R_2}\,\sigma_2^{R_2}(1-u)^{R_1}}{\Gamma(R_2+1)}\times E_{\rho_j,\lambda,q_j}^{\gamma_j,l_j,p_j}\left[z_1u^{\rho_1},......z_m u^{\rho_m}\right]du$$

$$=\frac{1}{\Gamma\delta}\sum_{R_1=0}^{\left[\frac{V_1}{U_1}\right]}\frac{(-V_1)_{U_1R_1}\,A_{V_1,R_1}\,\sigma_1^{R_1}}{\Gamma(R_1+1)}\sum_{R_2=0}^{\left[\frac{V_2}{U2}\right]}\frac{(-V_2)_{U_2R_2}\,A_{V_2,R_2}\,\sigma_2^{R_2}}{\Gamma(R_2+1)}\times \int_0^1 u^{\lambda-1}(1-u)^{\delta+R_1+R_2-1}\, E_{\rho_j,\lambda,q_j}^{\gamma_j,l_j,p_j}\left[z_1u^{\rho_1},......z_m u^{\rho_m}\right]du$$

$$=\Delta_{U_2,V_2,\sigma_2,R_2}^{U_1,V_1,\sigma_1,R_1}\times\frac{1}{\Gamma\delta}\int_0^1 u^{\lambda-1}(1-u)^{\delta+R_1+R_2-1}\, E_{\rho_j,\lambda,\,q_j}^{\gamma_j,l_j,p_j}\left[z_1u^{\rho_1},......z_m u^{\rho_m}\right]du$$

Using Theorem 1in above equation,we get

$$=\frac{\Gamma(\delta+R_1+R_2)}{\Gamma(\delta)}\Delta_{U_2,V_2,\sigma_2,R_2}^{U_1,V_1,\sigma_1,R_1}\, E_{\rho_j,\lambda+\delta+R_1+R_2,\,q_j}^{\gamma_j,l_j,p_j}\left[z_1,......z_m\right] \tag{3.3}$$

IV. SPECIAL CASES

4.1 *On setting* $U_1=U_2=2,\ A_{V_1,R_1}=(-1)^{R_1}$ *and* $A_{V_2,R_2}=(-1)^{R_2}$; *the Hermite polynomial in equations* (3.1), *we get the following result*:

Corollary (4.1)

$$\frac{\sigma_1^{\frac{V_1}{2}}\sigma_2^{\frac{V_2}{2}}}{\Gamma\delta}\int_0^1 u^{\lambda-1}(1-u)^{\delta+\frac{V_1}{2}+\frac{V_2}{2}-1}\, H_{V_1}\left(\frac{1}{2\sqrt{\sigma_1(1-u)}}\right)H_{V_2}\left(\frac{1}{2\sqrt{\sigma_2(1-u)}}\right)\times E_{\rho_j,\lambda,q_j}^{\gamma_j,l_j,p_j}\left[z_1u^{\rho_1},......z_m u^{\rho_m}\right]du$$

$$=\frac{\Gamma(\delta+R_1+R_2)}{\Gamma(\delta)}\sigma_1^{\frac{V_1}{2}}\sigma_2^{\frac{V_2}{2}}\, H_{V_1}\left(\frac{1}{2\sqrt{\sigma_1}}\right)H_{V_2}\left(\frac{1}{2\sqrt{\sigma_2}}\right)E_{\rho_j,\lambda+\delta+R_1+R_2,\,q_j}^{\gamma_j\,l_j,p_j}\left[z_1,......z_m\right] \tag{4.1}$$

4.2 *On setting* $U_1 = U_2 = 1,\ A_{V_1,R_1} = \frac{(V_1+1)_{R_1}}{(\Gamma R_1+1)(\Gamma R_1+1)}$ *and* $A_{V_2,R_2} = \frac{(V_2+1)_{R_2}}{(\Gamma R_2+1)(\Gamma R_2+1)}$; *the Bateman polynomial in equations* (3.1), *the following result is obtained after a little simplification*:

Corollary (4.2)

$$\frac{1}{\Gamma\delta}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1} Z_{V_1}[\sigma_1(1-u)]\, Z_{V_2}[\sigma_2(1-u)] E^{\gamma_j, l_j, p_j}_{\rho_j,\lambda,q_j}\left[z_1 u^{\rho_1},\ldots\ldots z_m u^{\rho_m}\right] du$$
$$= \frac{\Gamma(\delta+R_1+R_2)}{\Gamma(\delta)} Z_{V_1}[\sigma_1]\, Z_{V_2}[\sigma_2] E^{\gamma_j, l_j, p_j}_{\rho_j,\lambda+\delta+R_1+R_2, q_j}[z_1,\ldots\ldots z_m] \tag{4.2}$$

4.3 *On setting* $U_1 = U_2 = 1,\ A_{V_1,R_1} = \binom{V_1+\alpha}{V_1}\frac{1}{(\alpha+1)_{R_1}}$ *and* $A_{V_2,R_2} = \binom{V_1+\alpha}{V_1}\frac{1}{(\alpha+1)_{R_1}}$; *the* Laguerre Polynomials *in equations* (3.1), *the following result is obtained after a little simplification*:

Corollary (4.3)

$$\frac{1}{\Gamma\delta}\int_0^1 u^{\lambda-1}(1-u)^{\delta-1} L^{(\alpha_1)}_{V_1}[\sigma_1(1-u)]\ L^{(\alpha_2)}_{V_2}[\sigma_2(1-u)] E^{\gamma_j, l_j, p_j}_{\rho_j,\lambda,q_j}\left[z_1 u^{\rho_1},\ldots\ldots z_m u^{\rho_m}\right] du$$
$$= \frac{\Gamma(\delta+R_1+R_2)}{\Gamma(\delta)} L^{(\alpha_1)}_{V_1}[\sigma_1] L^{(\alpha_2)}_{V_2}[\sigma_2] E^{\gamma_j, l_j, p_j}_{\rho_j,\lambda+\delta+R_1+R_2, q_j}[z_1,\ldots\ldots z_m] \tag{3.7}$$

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REFERENCES

[1] A.A. Kilbas, M. Siago and R.K. Saxena, Generalized Mittag-Leffler function and generalized fractional calculus operators,Integral Transforms Spec. Funct. , Vol. **15**, (2004), pp. 31-49.

[2] M.K. Gujar, J.C. Prajapati and K. Gupta, A study of generalized Mittag Leffler function via Fractional Calculus, Journal of Inequalities and Special Functions, **5, No**. (2014), 6-13.

[3] G. M. Mittag-Leffler, Sur la nouvelle function $E_\alpha(x)$, C. R. Acad. Sci. Paris No **137** (1903), 554-558.

[4] T. R. Prabhakar,A singular integral equation with a generalized Mittag-Leffler function in the Kernel, Yokohama Math. J. No **19** (1971) 7-15. MR0293349 (45#2426).Zbl 0221.45003.

[5] T.O. Salim and A.W. Faraj, A generalization of Mittag-Leffler function and integral operator associated with fractional calculus, J. of Fract. Calc. and Appl., **3** (2012), 1-13.

[6] A.K. Shukla and J.C. Prajapati,On a generalization of Mittag-Leffler function and its properties, Math. Anal. Appl., **336** (2007), 797-811.

[7] H.M. Srivastava, A Contour inyegral involving Fox's H-function, Indian J.Math. **14**(1972), 1-6.

[8] H.M. Srivastava and H.L. Manocha, A Treatise on Generating Functions, Elli's Horwood Ltd. Chichester, John Wileyand Sons,New York 1984.

[9] G. Szego,Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ. Vol. **23** (1975), Fourth edition, Amer. Math. Soc.Providence, Rhode Island.

[10] A. Wiman, Uber de fundamental satz in der theorie der funktionen $E_\alpha(x)$, Acta Math. No. **29** (1905), 191-201.MR1555014.JFM36.0471.01.