Approximation Method for Hybrid Functional Differential Equations

S.N.SALUNKHE
DEPARTMENT OF MATHEMATICS
RANI LAXMIBAI MAHAVIDYALAYA PAROLA-425111 (INDIA)

ABSTRACT: *In this paper existence theorem for the Extremal solutions is proved for an initial value problem of nonlinear hybrid functional differential equations via constructive methods.*

Keywords: *Extremal solution, lower and upper solution, nonlinear hybrid functional differential equations, fixed point theorem*

INTRODUCTION

It is well known that the Banach contraction mapping principle is the only fixed point theorem in the nonlinear analysis which provides a useful method for approximating a unique solution for the initial and boundary value problems of ordinary differential equations via successive iteration. The best of our knowledge, there is no fixed point theorem or other method developed so far for hybrid functional differential equations without further assumptions on the nonlinearities involved in the equations.

In this paper, we using the ideas from Lakshmikantham and leela(2) and Ladde et.al(1). We establish the approximation results for extremal solutions of the hybrid functional differential equation between the given lower and upper solutions.

1. *STATEMENT OF PROBLEM*

Let $\mathbb{R}$ denote the real line and let $I_o = [-r, 0]$ and $I = [0, a]$ be two closed and bounded intervals in$\mathbb{R}$. Let$J = I_o \cup I$, then J is closed and bounded interval in $\mathbb{R}$. Let C denote the Banach space of all continuous real valued functions ϕ on I_o with supremum norm $\|.\|_C$ defined by

$\|\phi\|_C = \sup_{t\in J}|\phi\,(\text{t})|$ and $(x.y)(t) = x(t).y(t)$, $t \in J$ then C is a Banach algebra with this norm. Consider the Hybrid Functional Differential equation (In short HFDE)

$$\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t),x_t)}\right] = g(t, x(t), x_t) \quad \text{a.e} \quad t \in I$$

$$x(t) = \phi\,(\text{t}) \quad , \quad \text{t} \in I_o \tag{1.1}$$

Where $f: J \times \mathbb{R} \times C \to \mathbb{R} - \{0\}$, $g: J \times \mathbb{R} \times C \to \mathbb{R}$ and for each $\text{t} \in I$, $x_t: I_o \to C$ is continuous defined by $x_t(\theta) := x(t + \theta)$ for all $\theta \in I_o$.

By a solution of HFDE (1.1) we means a function $x \in C(J, \mathbb{R}) \cap AC(J, \mathbb{R}) \cap C(I_o, \mathbb{R})$ such that

i) The function $t \mapsto \frac{x}{f(t,x,x_t)}$ is absolutely continuous for each $x \in \mathbb{R}$ and

ii) x satisfies the equations in (1.1)

where $AC(J, \mathbb{R})$ denotes the space of absolutely continuous real valued functions defined on J.

The functional differential equations have been the must active area of research since long time; see Hale (8), Henderson (9) and the references there in. The study of functional differential equations in Banach algebra is very rare in the literature. Very recently, the study along this line has been initiated via fixed point theorem. See Dhage and Regan (4) and Dhage (3) and the references there in. The HFDE (1.1) is new to the literature and the study of this problem will definitely contribute immensely to the area of hybrid functional differential equation. The specialty of the results of the present paper lies in our HFDE (1.1) on$J$.

A function $f: J \times \mathbb{R} \times C \to \mathbb{R}$ is said to be long to the class $C(J \times \mathbb{R} \times C, \mathbb{R})$ of Caratheodory real valued functions defined on $J \times \mathbb{R} \times C$ if

i) $t \mapsto f(t, x, x_t)$ is measurable for each $x \in \mathbb{R}$ and

ii) $x \mapsto f(t, x, x_t)$ is continuous for each $t \in J$.

The following hypothesis concerning the function f is some time crucial in the study of HFDE (1.1)

(A_0) The function $x \mapsto \frac{x}{f(t,x,x_t)}$ is injective in$\mathbb{R}$. Hypothesis (A_0) holds in particular if the function $x \mapsto \frac{x}{f(t,x,x_t)}$ is increasing in $\mathbb{R}$.

We shall make use of the following result in what follows.

Lemma 1.1. Assume that hypothesis (A_0) holds. Further, if $g(\cdot, x(\cdot), x_t(\cdot)) \in L^1(J, \mathbb{R})$ then the HFDE (1.1) is equivalent to the hybrid integral equation (HIF)

$$x(t) = [f(t, x(t), x_t)] \left(\frac{x(t_0)}{f(t_0, x(t_0), x_{t_0})} + \int_{t_0}^{t} g(s, x(s), x_s) ds \right), \quad t \in J. \qquad 1.2$$

2. *METHODS OF LOWER AND UPPER SOLUTIONS*

In this section we prove an existence result for the HFDE (1.1) in a closed and bounded subset given by lower and upper solutions. A construction result is also obtained at the end of the section.

Definition 2.1. A function $u \in AC(J, \mathbb{R})$ is said to be a lower solution for HFDE (1.1) if

i) $t \mapsto \frac{u(t)}{f(t,u(t),u_t)}$ is almost everywhere continuous for each $x \in \mathbb{R}$ and

ii) $\frac{d}{dt}\left[\frac{u(t)}{f(t,u(t),u_t)}\right] \le g(t, u(t), u_t)$ a.e $t \in J$, $u(t_0) \le x(t_0)$

and a function $v \in AC(J, \mathbb{R})$ is said to be a lower solution for HFDE (1.1) if

i) $t \mapsto \frac{v(t)}{f(t,v(t),v_t)}$ is almost everywhere continuous for each $x \in \mathbb{R}$ and

ii) $\frac{d}{dt}\left[\frac{v(t)}{f(t,v(t),v_t)}\right] \ge g(t, v(t), v_t)$ a.e $t \in J$, $v(t_0) \ge x(t_0)$

A solution of the HFDE (1.1) is a lower as well as upper solution and vice versa.

If the existence of lower and upper solutions of HFDE (1.1) such that $u(t) \le v(t)$, $t \in J$ then we prove the existence of a solution of the HFDE(1.1) in the closed set $\overline{\Omega} = \{x \in E: u(t) \le x \le v(t), \ t \in J\}$. We study the problem in the space $C(J, \mathbb{R})$ of continuous real valued functions defined on J.

We need the following hypotheses in follows.

(A_1) The function $x \mapsto \frac{x}{f(t,x,x_t)}$ is increasing in $\mathbb{R}$ almost everywhere

(A_2) There exists a constant $L > 0$ such that $|f(t,x,x_t) - f(t,y,y_t)| \leq L|x - y|$ for all t $\in J$ and $x, y \in \mathbb{R}$.

(A_3) There exists a function $h \in L^1(J,\mathbb{R})$ such that $|g(t,x,x_t)| \leq h(t)\ a.e.$ t $\in J$ for all $x \in \overline{\Omega}$.

Theorem: 2.1. Let $u, v \in C(J,\mathbb{R})$ be lower and upper solutions of HFDE (1.1) satisfying $u(t) \leq x \leq v(t)$, t $\in J$ and let the hypotheses (A_1)-(A_3) hold. Suppose also that the condition that

$$L\left(\left|\frac{x(t_0)}{f(t_0,x(t_0),x_{t_0})}\right| + \|h\|_{L^1}\right) < 1 \qquad 2.1$$

is satisfied. Then there exists a solution $x(t)$ of HFDE (1.1) in the closed set $\overline{\Omega}$ that is

$$u(t) \leq x(t) \leq v(t), \text{t} \in J.$$

Proof. Define a function $\rho: J \times \mathbb{R} \to \mathbb{R}$ by $\rho(t,x) = max\{u(t), \min x(t), v(t)\}$ 2.2

Then $\bar{g}(t,x,x_t) = g(t,\rho(t,x),x_t)$ defines a continuous extension of g on $J \times \mathbb{R} \times C \to \mathbb{R}$ satisfying

$|\bar{g}(t,x,x_t)| = |g(t,\rho(t,x),x_t)| \leq h(t)\ a.e.$t $\in J$ for all $x \in \mathbb{R}$.

Hence by Dhage and Lakshmikantham (3), the HFDE

$$\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t),x_t)}\right] = \bar{g}(t,x(t),x_t) \text{ a.e } \text{ t} \in J$$

$$x(t_0) = x_0 \in \mathbb{R} \qquad 2.3$$

has a solution x on J.

For $\varepsilon > 0$, let

$$\frac{u_\varepsilon(t)}{f(t,u_\varepsilon(t),(u_\varepsilon)_t)} = \frac{u(t)}{f(t,u(t),u_t)} - \varepsilon(1+t) \qquad 2.4$$

and

$$\frac{v_\varepsilon(t)}{f(t,v_\varepsilon(t),(v_\varepsilon)_t)} = \frac{v(t)}{f(t,v(t),v_t)} + \varepsilon(1+t) \qquad 2.5$$

for t $\in J$. Then in view of hypotheses (A_1), we obtain

$$u_\varepsilon(t) \leq u(t) \text{ and } v(t) \leq v_\varepsilon(t) \quad \text{for t} \in J \qquad 2.6$$

. Since $u(t_0) \leq x_0 \leq v(t_0)$

one has $u_\varepsilon(t_0) \leq x_0 \leq v_\varepsilon(t_0)$ 2.7

Next, we show that

$$u_\varepsilon(t) \leq x(t) \leq v_\varepsilon(t) \qquad \text{for} \in J \text{ t.} \qquad 2.8$$

Define $X(t) = \frac{x(t)}{f(t,x(t),x_t)}$, t $\in J$,

define $U_\varepsilon(t) = \frac{u_\varepsilon(t)}{f(t,u_\varepsilon(t),(u_\varepsilon)_t)}$, $U(t) = \frac{u(t)}{f(t,u(t),u_t)}$

and $V_\varepsilon(t) = \frac{v_\varepsilon(t)}{f(t,v_\varepsilon(t),(v_\varepsilon)_t)}$, $V(t) = \frac{v(t)}{f(t,v(t),v_t)}$

for all $t \in J$.

Take $\exists\, t_1 \in (t_0, t_0 + a)$ such that $x(t_1) = v_\varepsilon(t_1)$ and $u_\varepsilon(t) \leq x(t) \leq v_\varepsilon(t)$, $t_0 \leq t \leq t_1$ if (2.8) is not true,

If $x(t_1) > v(t_1)$ then $\rho(t_1, x(t_1)) = v(t_1)$. Morever, $u(t_1) \leq \rho(t_1, x(t_1)) \leq v(t_1)$.

Now $V'(t_1) \geq g(t_1, v(t_1), v_{t_1}) = \bar{g}(t_1, x(t_1), x_{t_1}) = X'(t)$ for all $t \in J$.

Since $v_\varepsilon(t) > V'(t)$ for all $t \in J$

We have that

$$V_\varepsilon'(t_1) > X'(t_1) \tag{2.9}$$

However, $X(t_1) = v_\varepsilon(t_1)$

and $X(t) = v_\varepsilon(t)\,, \quad t_0 \leq t \leq t_1$

together imply that

$$\frac{X(t_1+h) - X(t_1)}{h} > \frac{V_\varepsilon(t+h) - V_\varepsilon(t)}{h} \quad \text{if } h < 0 \text{ small}$$

Taking the limit as $h \to 0$ in the above inequality yields

$$X'(t_1) \geq V_\varepsilon'(t_1)$$

which is contradiction to (2.9), hence

$$X(t) = v_\varepsilon(t) \quad \text{for all } t \in J$$

Consequently,

$$u_\varepsilon(t) \leq x(t) \leq v_\varepsilon(t)\,, t \in J$$

letting the limit as $\varepsilon \to 0$ in the above inequality, we obtain

$$u(t) \leq x(t) \leq v(t), \; t \in J.$$

This completes the proof.

The existence of lower and upper solutions is an essential ingredient in many problems of nonlinear functional differential equations and which do exist for every functional differential equation obviously. The following simple results give the sufficient conditions that guarantee the existence of lower and upper solutions for the HFDE (1.1) on J.

We consider the following hypotheses

(A_4) The function $x \mapsto g(t, x, x_t)$ is no increasing in $\mathbb{R}$ for almost everywhere $t \in J$.

Theorem: 2.2. Suppose that hypotheses $(A_1) - (A_4)$ hold. Further if the condition (2.1) is satisfied, then there exists a lower solution u_0 and upper solution v_0 for the HFDE (1.1) such that $u_0(t) \leq v_0(t)$ on J.

Proof. Let $y(t)$ be the unique solution of the HFDE

$$\frac{d}{dt}\left[\frac{y(t)}{f(t,y(t),y_t)}\right] = \bar{g}(t, x(t), x_t) \text{ a.e. } t \in J$$

$$y(t_o) = x_0 \in \mathbb{R} \qquad 2.10$$

which does exists in view of condition (2.1)

Define

$$\frac{u_0(t)}{f(t,u_0(t),(u_0)_t)} = \frac{y(t)}{f(t,y(t),y_t)} - R_0\,, t \in J \qquad 2.11$$

and

$$\frac{v(t)}{f(t,v(t),(v_0)_t)} = \frac{y(t)}{f(t,y(t),y_t)} + R_0\,, t \in J \qquad 2.12$$

For some real number $R_0 > o$, choose R_0 so large that

$$\frac{u_0(t)}{f(t,u_0(t),(u_0)_t)} \leq 0 \leq \frac{v(t)}{f(t,v(t),(v_0)_t)}, \quad t \in J$$

and \ $$\frac{v(t)}{f(t,v(t),(v_0)_t)} = \frac{y(t)}{f(t,y(t),y_t)} \geq g(t, u_0(t), (u_0)_t), \; t \in J$$

Also from (2.11) and (2.12), it follows that

$$u_0(t_0) \leq x_0 \leq v_0(t_0)$$

Thus the function $u_0(t)$ and $v_0(t)$ are respectively the desired lower and upper solutions for the HFDE(1.1) on J. Finally, from the hypotheses (A_1) and the inequality (2.13) it follows that $u_0(t) \leq v_0(t)$ and this completes the proof.

REFERENCES

[1] G.S.Ladde,V.Lakshmikantham and A.S.Vatsala,Monotone iterative techniques for nonlinear differential equations,Pitman, New York, 1985.

[2] V.Lakshmikantham and S.Leela, Differential and Integral Inequalities, Academic press,Newyork,1969

[3] B.C.Dhage, A fixed point theorem in Banach algebras with applications to functional integral equations,Kyungpook Math.J.44(2004),145-155.

[4] B.C.Dhage and V.Lakshmikantham, Global existence and convergence theorem for hybrid differential equations (Submitted for publication)

[5] B. C. Dhage, S. N. Salunkhe, Ravi Agarwal and W. zhang, A functional differential equations in Banach algebras, Math. Inequ. Appl. 8(1) (2005), 89-99.

[6] B. C. Dhage, S. N. Salunkhe and R. U. Verma, A neutral functional differential equation in Banach Algebras. Math. Sci. Resarch. J. 9(4) (2005), 101-113.

[7] Salunkhe S. N., Timol M. G. – Positivity Results ForFunctional Differential Equations on Unbounded Intervals.VNSGU Journal of Science and Technology V1(2)- (2009),161-168.

[8] J.K.Hale,Theory of functional differential equations, Springer Verlag,New York, 19677.

[9] J.Henderson, Boundary value problems for functional differential equations, World Scientific, Singapore, 1995.