An Extension of the Paper "Maximum Distance in Graphs"

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Abstract

Thamarai Selvi and Vaidyanathan [1] introduced the concept of Maximum distance (M - distance) in a graph. They claimed a few relations between the (usual) metric distance and M - distance related to a graph. Further, they defined M - Eccentricity, M - Radius, M - Diameter and M - Denter related to a graph and obtained them for a few standard graphs.

In this paper, we made improvements for a considerable number of results of the above mentioned paper.

Our paper is divided into three sections. The first one deals with introduction and basic results; the middle one for the calculation of $d^{M}(u, v)$ (*M*-distance) for all pairs of vertices *u*, *v* of the graphs K_n , S_n and $K_{m,n}$ for suitable *m*, *n* and to P_2 , P_3 . The last section is devoted with the calculations of M-Eccentricity etc. of the above mentioned graphs.

Keywords: M-distance, Maximum radius, Maximum diameter

I. Introduction and Basic Results:

ThamaraiSelvi and Vaidyanathan [1] introduced the concept of Maximum distance (M-distance) in a graph by considering the length of any shortest path between any two vertices, the sum of the degrees of all the vertices in the path in addition to the total number of vertices in the path.

Mathematically, if u and v are any two vertices of a graph G, then the M-distance between u and v in G, denoted by $d^{M}(u, v)$ is defined to be

Min. $\{l^{M}(P) : P \text{ is any u-v path in } G\},\$

where,
$$l^{M}(P) = d(u, v) + \sum_{x \in V(P)} \deg_{G}(x) + \sum_{w \in P(G)} \deg_{G} |w|$$

'd' denotes the usual metric distance function on the set vertex V(G) of G, V(P) denotes the set of all vertices of G that are in the path P between u and v, including u as well as v and the last sum denotes the number of vertices of G that are in P (Observe that the last sum in the above expression is confusive, a better way is to denote it by O(P), the number of elements in P).

Clearly, it follows that $d^{M}(u, v) \ge d(u, v)$ for all pairs of vertices u, v in G.

Remark (1.1) They claimed that the above inequality is an equality iff (if and only if) u and v are the same.

But, if v = u, then P consisting of the single vertex u, i.e., $P = \{u\}$ is the shortest

u – u path in G. So

$$d^{M}(u, u) = e^{M}(P) = d(u, u) + \sum_{x \in V(P)} \deg_{G}(x) + \sum_{w \in P(G)} \deg_{G} |w|$$

 $= 0 + \deg u + 1$ (there is only vertex u in P)

$$\geq 1$$

 $\Rightarrow d^{M}(u, u) \neq 0 = d(u, u).$

So d^M is not a metric on V(G).

Their **Theorem 1** is the following: If G is a connected graph, then the M – distance is a metric on V(G).

Now, we redefine M – distance on V(G) as follows:

Definition (1.2): G is a non-empty, finite, simple and connected graph; M is a positive integer. The M-distance function d^M on V(G) is a mapping V(G) x V(G) $\rightarrow R^+$ (the set of all non-negative reals) where, for all u, $v \in V(G)$,

$$d^{M}(u,u) = \begin{cases} 0 \text{ if } v = u \\ \min \{l^{M}(P), P \text{ being any } u - v \text{ path in } G \text{ and } l^{M}(P) = d(u, u) + \sum_{x \in V(P)} \deg_{G}(x) + O(P) \text{ if } v \neq u \end{cases}$$

Now, by definition, the new d^M is such that $d^M(u, u) = 0$ and $u, v \in V(G)$ with $v \neq u$ implies that

$$d^{M}(u, v) = d(u, v) + ... > 0.$$

Thus, $d^{M}(u, v) \ge 0$ and = 0 iff v = u.

Clearly $d^{M}(u, v) = d^{M}(v, u)$ for all u, v in V(G).

Now, we show that the triangular inequality holds. (The argument given in their Theorem needs modification).

Let u, w, v be in V(G) and P, Q be shortest u - w and w - v paths respectively in G. So follows that $d^M(u, w) = l^M(P)$ and $d^M(w, v) = l^M(Q)$. Observe that P U Q need not be a u - v path in G. It is a trail. Let R be a u - v path in P U Q. Since the set R is a subset of the set P U Q, it follows that

$$\begin{split} d^{M}(u, v) &= e^{M}(R) \leq l^{M}(P) + l^{M}(Q) \\ &= d^{M}(u, w) + d^{M}(w, v). \end{split}$$

Thus, d^{M} is a metric on V(P).

Remark 1.3. In their proposition (1.2), they claimed the following. In a connected graph G, two distinct vertices u, v are adjacent $\Leftrightarrow d^M(u, v) = \deg u + \deg v + 3$.

Their argument for the first part is alright. u and v are adjacent in G \Rightarrow P= {u, v} (u \rightarrow v) is the shortest u – v path in G \Rightarrow $l^{M}(P) = d(u, u) + \sum_{x \in V(P)} \deg_{G}(x) + O(P)$

 $= 1 + \deg u + \deg v + 2 = \deg u + \deg v + 3.$

The proof of the second part needs a little bit modification.

Let $u, v \in V(G)$ be such that $d^{M}(u, v) = degu + degv + 3$.

We claim that u and v are adjacent in G. Suppose not; any shortest u - v path P in G contains atleast three vertices, the end vertices being u and v. Now $d^M(u, v) = d(u, v) + \sum_{x \in W(P)} \deg_G(x) + O(P) \ge 2 + \{\deg u + \deg v\} + 3$

$$= \deg u + \deg v + 5$$

which is a contradiction.

Hence, the conclusion follows.

2. Calculation of $d^{M}(u, v)$ for all ordered pairs(u, v) of vertices u, v of a graph.

Observation (2.1): Let G be a non-empty, finite, simple and connected graph. (Here under, we consider only such graphs). So the number of vertices 'n' of G is atleast 2.

If u, v ε V(G) and v = u then d^M (u, v) = d^M(u, u) = 0. This happens for all u ε V(G). Thus d^m takes the value '0' at all the n pairs (u, u) of G.

Now, let $V(G) = \{ v_1, v_2, ..., v_n \}$. Now we have to calculate $d^M(v_i, v_j)$ for

 $1 \le i < j \le n$, since d^M is symmetric. So the number of distinct positive values we get for $d^M(u, v)$ is given by $d^M(v_i, v_j)$ for i ϵ {1, ..., n-1} and j ϵ {(i+1), ..., n}. Thus, the number is

$$\sum_{i=1}^{n-1} 1\left\{\sum_{j=i+1}^{n} 1\right\} = \sum_{i=1}^{n-1} \{n - (i+1) + 1\}$$
$$= \sum_{i=1}^{n-1} (n-i) = n(n-1) - \frac{(n-1)n}{2}$$
$$= \frac{n(n-1)}{2}$$

So, the number of distinct values that $d^{M}(v_{i}, v_{j})$ $(1 \le i \le j \le n)$ takes is $1 + \frac{n(n-1)}{2} = \frac{(n^{2} - n + 2)}{2}$, one value being

'0'.

Remark (2.2): For any graph G, $d^{M}(u, u) = 0$ for all vertices u of G.

Theorem 2.3. For the complete graph K_n ($n \ge 2$) $d^M(u, v) = (2n+1)$ for any (distinct) pair of vertices u, v of K_n .

Proof. Let u, v be distinct vertices of K_n . Clearly u and v re adjacent in K_n and deg u = deg v = (n - 1). Now, by Remark (1.3) $d^M(u, v) = degu + degv + 3 = (n - 1) + (n - 1) + 3 = (2n + 1)$.

Remark (2.4): The path $P_2 = K_2$ and it is a special case of Theorem(2.3).

Theorem (2.5): For the path P₃

 $d^{M}(u, v) = \begin{cases} 6 & if exactly one of u, v is not an end vertices of P_{3} \\ 9 & if u and v are the end vertices of P_{3} \end{cases}$

Proof. Let $V(P_3) = \{v_1, v_2, v_3\}$. Now $deg(v_1) = deg(v_3) = 1$ and $deg(v_2) = 2$.

If $u = v_1$ and $v = v_2$ or $u = v_2$ and $v = v_3$, then u and v are adjacent in P₃. So by Remark(1.3)

 $d^{M}(v_{1}, v_{2}) = degv_{1} + degv_{2} + 3 = 1 + 2 + 3 = 6$

 $d^{M}\left(v_{2},\,v_{3}\right)=\ degv_{2}+\ degv_{3}\ +\ 3=2+1+\ 3=6$

Let $u = v_1$ and $v = v_3$. Now $P = \{v_1, v_2, v_3\}$ is the only shortest path in P_3 .

So, by definition

 $d^{M}(v_{1}, v_{3}) = d(v_{1}, v_{3}) + \sum_{x \in \{v_{1}, v_{2}, v_{3}\}} deg \ x + O(P)$ = 2 + deg v_{1} + deg v_{2} + deg v_{3} + 3 = 2 + (1+2+1) + 3 = 9.

Remark (2.6): The star graph $S_1 = K_2$ and $S_2 = P_3$ and these are already discussed (Remark (2.4) and Theorem (2.5)).

Theorem (2.7). For the star graph S_n ($n \ge 3$) with vertex set $V(S_n) = \{u_0, v_1, v_2, ..., v_n\}$, where u_0 is the centre (adjacent with all v's) and v_i (i = 1, 2, ..., n) are such that they are (adjacent with all v_i) and v_i (i = 1, 2, ..., n) are adjacent with u_0 only, then

 $d^{M}(u_{0}, v_{i}) = (n + 4)$ for i = 1, 2, ..., n

and

 $d^{M}(v_{i}, v_{j}) = (n + 7)$ for $1 \le i \ne j \le n$

Proof. Clearly, $deg(u_0) = n$, $deg(v_i) = 1$ for all i.

Since u_0 is adjacent with every v_i . by Remark (1.3)

 $d^{M}(u, v_{i}) = \deg u_{0} + \deg v_{i} + 3 = n + 1 + 3 = (n + 4)$ for all v_{i} .

Let v_i , v_j be distinct vertices of S_n .

Since, d^M is symmetric, without loss of generality, we can assume that $1 \le i < j < n$. The shortest $v_i - v_j$ path is $P : v_i \rightarrow u_0 \rightarrow v_j$ and it is of length 3. Now, by definition

$$d^{M}(v_{i}, v_{j}) = d(v_{i}, v_{j}) + \sum_{x \in P\{v_{i}, u_{0}, v_{j}\}} \deg x + O(\{v_{i}, u_{0}, v_{j}\})$$

= 2 + (deg v_{i} + deg u_{0} + deg v_{j}) + 3
= 2 + (1 + n + 1) + 3 = (n + 7).

This completes the proof of the theorem.

Remark 2.8. The complete bipartite graph $K_{1,1} = K_2$, $K_{1,2} = S_2$ and $K_{1,n}$ ($n \ge 3$) is S_n and these are already discussed.

Now, we consider the following.

Theorem (2.9). For the complete bipartite graph $K_{m,n}$ (m, $n \ge 2$) with (vertex) bipartition (X, Y), where $X=\{u_1, u_2, ..., u_m\}$ and $Y = \{v_1, v_2, ..., v_n\}$,

$$d^{M}(u_{i}, u_{i'}) = (2n + m + 5) \quad 1 \le i < i' \le m,$$

$$d^{M}(v_{j}, v_{j'}) = (2m + n + 5) \quad 1 \le j < j' \le n$$

and

 $d^M(u_i,\,v_j) = (n+m+3): \ \ i \; \epsilon \; \{1,2,\ldots\,m\} \ \, and \; j \; \epsilon \; \{1,2,\ldots,n\}$

Proof. By the definition of $K_{m,n}$, clearly $deg(u_i) = n$ and $deg(v_i) = m$ for i $\epsilon \{1, 2, ..., m\}$ and j $\epsilon \{1, 2, ..., n\}$.

Let i, i¹ ε {1,2,...,m} and i¹ \neq i. Without loss of generality we can assume that $1 \le i < i' \le m$.

Since u_i is connected with $u_{i'}$ by means of any vertex $v \in Y$, it follows that $P = \{u_i, v, u_{i'}\}$ is a shortest $u_i - u_{i'}$ path in $K_{m,n}$. So, by definition

$$d^{M}(u_{i}, u_{i'}) = d(u_{i}, u_{i'}) + \sum_{x \in P\{v_{i}, u_{0}, v_{j}\}} \deg x + O(\{v_{i}, u_{0}, v_{j}\})$$

= 2 + (deg u_{i} + deg v + deg u_{i'}) + 3
= 2 + (n + m + n) + 2
= (2n + m + 5).

Similarly, it follows that

$$d^M(v_j, v_j) = 2m + n + 5.$$

Since u_i and v_j are adjacent in $K_{m,n}$, by Remark (1.3)

$$d^{M}(u_{i}, v_{j}) = \deg u_{i} + \deg v_{j} + 3$$
$$= n + m + 3.$$

§3. Calculation of M-Eccentricity etc for Standard Graphs

Thamarai Selvi and Vaidyanathan introduced these concepts for standard graphs as follows

As usual G and H have their usual meanings.

Definition (3.1) a) Let $v \in V(G)$. The M – Eccentricity of v, denoted by $e^{M}(v)$ is defined to be max { $d^{M}(u,v)$: $u \in V(G)$ }

If $u_0 \in V(G)$ be such that $d^M(u_0, v) = e^M(v)$, then u_0 is called an M – Eccentric vertex of v. Further, it is also called an M – Eccentric vertex of G.

b) The maximum radius called the M – radius of G, dented by $r^{M}(G) = \min \{e^{M}(u,v): v \in V(G)\}$.

c) The maximum diameter called the M – diameter of G, denoted by $d^{M}(G) = \max\{e^{M}(v): v \in V(G)\}$

d) The maximum centre called the M – centre of G, denoted by $C^{D}(G)$ is the subgraph of G induced by the set of all vertices of minimum M – eccentricity.

e) G is said to be M – self centered iff $C^{M}(G) = G$ or equivalently $r^{M}(G) = d^{M}(G)$.

f) The set of all vertices of maximum M – eccentricity is called the M – periphery of G.

Result (3.2). For the complete graph K_n $(n \ge 2)$ $r^M(K_n) = d^M(K_n) = (2n + 1)$, $C^D(K_n) = K_n \implies K_n$ is self centered and M – periphery of K_n is $V(K_n)$.

Proof. Th(2.1) gives that $d^{M}(u, v) = (2n + 1)$ for all $u, v \in V(K_n)$. So $e^{M}(v)$ is the same and = (2n + 1) for all $v \in V(G)$. Hence the result follows from Definition (3.1). *Result* (3.3). For the paths P_n (n = 2,3), $r^M(P_2) = d^M(P_2) = 5$ and $r^M(P_3) = 2(3) = 6$ and $d^M(P_3) = 9 = 4(3) - 3$.

Proof. Th (2.2) (a) gives that $d^{M}(u,v) = 5$

Since $P_2 = K_2$, the result follows from the above Result when n = 2.

Suppose n = 3. From Th.(2.5), we get that $r^{M}(P_{3}) = \min \{6, 9\} = 6 = 2(3)$.

 $d^{M}(P_{3}) = \max \{6, 9\} = 9 = 4(3) - 3.$

If $P_3 = v_1 - v_2 - v_3$, then $e^M(v_1) = 9 = e^M(v_3)$, $e^M(v_2) = 6$.

So $C^{D}(P_{3})$ is the subgraph of P_{3} consisting of the single vertex v_{2} and the M – periphery of P_{3} is the vertex set consisting of the vertices v_{1} and v_{3} , the edge set being the empty set.

Result (3.4). For the star graph S_n ($n \ge 3$), $r^M(S_n) = (n + 4)$, $d^M(S_n) = (n + 7)$.

 $C^{D}(S_{n})$ is the subgraph generated by the single vertex u_{0} and the periphery is the set $\{v_{1}, v_{2}, ..., v_{n}\}$.

 $e^{M}(u_{0}) = Max \{d^{M}(u_{0}, v_{i}): i = 1, 2, ..., n\} = (n + 4)$

For any $i \in \{1, 2, ..., m\}$

 $e^{M}(v_{i}) = Max (d^{M}(v_{i}, u_{0}), Max\{d^{M}(v_{i}, v_{j}): j \in \{1, 2, ..., n\} - \{i\}\}$

$$=$$
 Max (n + 4, n + 7)

$$=(n+7)$$

So $r^{M}(S_{n}) = Min \{e^{M}(u_{0}), \{e^{M}(v_{i}): i=1,2,...,n\}\}$

$$=$$
 Min (n+4, n+7)

= n + 4

 $d^{M}(S_{n}) = Max \ \{e^{M}(u_{0}), \ Max\{e^{M}(v_{i}): i = 1, 2, ..., n\}$

$$=$$
 Max {n+4, n+7} = (n + 7)

So $C^{D}(S_{n})$ is the subgraph with single vertex set u_{0} and the M – periphery of $C^{D} S_{n}$ is the vertex set $\{v_{1}, v_{2}, ..., v_{n}\}$

Result (3.5). For the complete bipartite graph $K_{m,n}$ (m, $n \ge 2$) and $m \le n$,

 $r^{M}(K_{m,n}) = 2m + n + 1$, $d^{M}(K_{m,n}) = m + 2n + 1$, $C^{D}(K_{m,n})$ is the subgraph generated by Y and M – periphery is the set X.

Proof. By Th. (2.3), for any $u \in X$

 $e^{M}(u) = Max \{ d^{M}(u, z) : z \in V(K_{m,n}) = X \cup Y \}$

- = Max (Max { $d^{M}(u, u')$: $u' \in X$ }, Max{ $d^{M}(u, v)$: $v \in Y$ })
- = Max (m + 2n + 5, m + n + 3)

$$= (m + 2n + 5)$$

For any $v \in Y$

$$e^{M}(v) = Max \{ d^{M}(v, z): z \in X \cup Y \}$$

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- = Max (Max { $d^{M}(v, u)$: $u \in X$ }, Max{ $d^{M}(v, v')$: $v' \in Y$ })
- = Max (m + n + 3, 2m + n + 5)
- = (2m + n + 5).

Without loss of generality, we can assume that $m \le n$. Now,

 $r^{M}(K_{m,n}) = Min \{m + 2n + 5, 2m + n + 5\}$

= 2m + n + 5

 $d^{M}(K_{m,n}) = Max \{m + 2n + 5, 2m + n + 5\}$

$$= m + 2n + 5$$

So $C^{D}(K_{m,n})$ is the subgraph generated by the set $Y = \{v_1, v_2, ..., v_n\}$. This is null set with n vertices.

The M – Periphery of $K_{m,n}$ is the vertex set $X = \{u_1, u_2, ..., u_m\}$.

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