Symmetry Reductions of $(U_T + U^3U_X + \alpha U_{XXX})_X + \beta U_{YY} = 0$

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Abstract. *A (2+1)-dimensional generalized KdV equation $(u_t+u^3u_x+\alpha u_{xxx})_x+\beta u_{yy}=0$, α, $\beta\in R^+$ is subjected to Lie's classical method. Classification of its symmetry algebra into one- and two-dimensional subalgebras is carried out in order to facilitate its systematic reduction to (1+1) dimensional PDE and then to ODEs. A solution containing two arbitrary functions of time t is also determined.*
Key words. A (2+1)-dimensional generalized KdV equation; Symmetry algebra.

1. introduction

The KdV equation

(1.1) $$u_t + uu_x + \delta u_{xxx} = 0,$$

is integrable in the sense that it possesses solitons, B¨acklund transformations, Lax pair, infinite number of conservation laws and Painlev´e property. Whitham [7] has given a representation of a periodic wave as a sum of solitons for (1.1). Miura [4] established a relation between (1.1) and the modified KdV equation $u_t+au^2u_x+u_{xxx}=0$.

Liu and Yang [3] studied bifurcation problems for a generalized KdV equation

(1.2) $$u_t + au^n u_x + u_{xxx} = 0, \qquad n \geq 1, \quad a \in R.$$

A generalized version of (1.2) in the form

(1.3) $$u_t + u^n u_x + \alpha(t)u + \beta(t)u_{xxx} = 0,$$

has recently been studied for its symmetry group and similarity solution by Senthilkumaran, Pandiaraja and Mayil Vaganan [6].

In this paper we introduce yet another (2 + 1)-dimensional variable coefficient KdV equation

(1.4) $$(u_t + u^3u_x + \alpha u_{xxx})_x + \beta u_{yy} = 0, \alpha, \beta \in R^+.$$

Our intention is to show that equation (1.4) admits a five-dimensional Lie algebra, and classify it into the one- and two-dimensional sub algebras in order to reduce (1.4) to (1+1)-dimensional partial differential equations (PDEs) and then to ordinary differential equations (ODEs). It is shown that (1.4) reduces to a once differentiated generalized KdV equation, a linear equation $W_{rr}(r,s) = 0$. We shall establish that the symmetry generators form a closed Lie algebra and this allowed us to use the recent method due to Ahmad, Bokhari, Kara and Zaman [1] to sucessively reduce (1.4) to (1+1)-dimensional PDEs and ODEs with the help of two-dimensional Abelian and solvable non-Abelian sub algebras.

This paper is organised as follows: In section 2 we determine the symmetry group of (1.4) and write down the associated Lie algebra. In section 3 we consider all one dimensional sub algebras and obtain the corresponding reductions to (1+1)-dimensional

PDEs. In section 4 we show that the generators form a closed Lie algebra and use this fact to reduce (1.4) successively to (1+1)-dimensional PDEs and ODEs. Section 5 summarises the results of the present work.

2. The symmetry group and Lie algebra of (1.4)

If (1.4) is invariant under classical Lie group of infinitesimal transformations (Olver [5], Blumen and Kumei [2])

(2.1) $$x_i^* = x_i + \epsilon\xi_i(x,y,t,u) + O(\epsilon^2), i = 1,2,3,4,$$

where $\xi_1 = \xi$, $\xi_2 = \eta$, $\xi_3 = \tau, \xi_4 = \phi$, then the fourth prolongation $pr^{(4)}V$ of the corresponding vector field

(2.2) $$V = \tau(x,y,t;u)\partial_t + \xi(x,y,t;u)\partial_x + \eta(x,y,t;u)\partial_y + \phi(x,y,t;u)\partial_u.$$

satisfies

(2.3) $$pr^{(4)}V\,\Omega(x,y,t;)|_{\Omega(x,y,t;u)} = 0.$$

The determining equations are obtained from (2.3) and solved for the infinitesimals ξ,η,τ,ϕ

(2.4) $$\xi = c_1 + \frac{c_5 x}{3} - \frac{(c_2 y)}{2\beta} \quad \eta = c_3 + c_2 t + \frac{2c_5 y}{3},\ \tau = c_4 + c_5 t,\ \emptyset = \frac{-2c_5 u}{9}.$$

Now we write down the five symmetry generators corresponding to each of the constants c_i, i =1,2,...,5 involved in the infinitesimals, viz.,

(2.5) $$V_1 = \partial_x,\ V_2 = \frac{-y}{2\beta}\partial_x + t\partial_y,\ V_3 = \partial_y,\ V_4 = \partial_t,$$
$$V_5 = \frac{x}{3}\partial_x + \frac{2y}{3}\partial_y + t\partial_t - \frac{2u}{9}\partial_u\,.$$

These symmetry generators form a closed Lie algebra as is seen from the following commutator Table:

Table 1

$[V_i,V_j]$	V_1	V_2	V_3	V_4	V_5
V_1	0	0	0	0	$V_1/3$
V_2	0	0	$V_1/2\beta$	$-V_3$	$-V_2/3$
V_3	0	$-V_1/2\beta$	0	0	$2V_3/3$
V_4	0	V_3	0	0	V_4
V_5	$V_1/3$	$V_2/3$	$2V_3/3$	V_4	0

3. Reductions of (1.4) by one-dimensional sub algebras

As there are five generators, we consider the reductions of (1.4) under each generator separately.

Case 1. Sub algebra $L_{s,1} = \{V_1\}$

The characteristic equation associated the generator V_1 is

(3.1) $$\frac{dx}{1} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{0}.$$

and the similarity transformation is

(3.2) $$s = y,\ r = t,\ u = W(r,s).$$

Using (3.2) in (1.4), the latter changes to a linear PDE

(3.3) $$W_{ss} = 0,$$

whose general solution

(3.4) $$W(r,s) = A(r)s + B(r),$$

yields a solution of (1.4) involving two arbitrary functions $A(t)$ *and* $B(t)$.

Case 2. Sub algebra $L_{s,2}=\{V_2\}$

The characteristic equation and its solutions are

(3.5) $$\frac{dx}{y}=\frac{dy}{-2\beta t}=\frac{dt}{0}=\frac{du}{0},$$

(3.6) $$s=2\beta tx+\frac{y^2}{2},\ \ r=t,\ \ u=W(r,s).$$

In view of (3.6), (1.4) transforms into

(3.7) $$(2sW_s+4\beta r^2W^3W_s+2rW_r\ +W\ +16\alpha\beta^3r^4W_{sss})_s\ =0.$$

Case 3. Sub algebra $L_{s,3}=\{V_3\}$

The characteristic equation associated with V_3 is

(3.8) $$\frac{dx}{0}=\frac{dy}{1}=\frac{dt}{0}=\frac{du}{0}.$$

Integrating (3.8) we get

(3.9) $$s=x,\ \ r=t,\ \ \ u=W(r,s).$$

Equations (1.4) and (3.9) together lead to

(3.10) $$(W_r\ +W^3W_s+\alpha W_{sss})_s\ =0.$$

The above equation is once differentiated KdV equation with variable coefficient.

Case 4. Sub algebra $L_{s,4}\ =\{V_4\}$

The characteristic equation associated with V_4 is

(3.11) $$\frac{dx}{0}=\frac{dy}{0}=\frac{dt}{1}=\frac{du}{0}.$$

Integration of (3.11) gives rise to

(3.12) $$s=y,\ \ r=x,\ \ u=W(r,s).$$

Inserting the ansatz (3.12) into (1.4), we find that

(3.13) $$(W^3W_r\ +\alpha W_{rrr})_r\ +\beta W_{ss}\ \ =0.$$

Case 5. Sub algebra $L_{s,5}=\{V_5\}$

The solution of the characteristic equation associated with V_5, namely,

(3.14) $$\frac{dx}{x}=\frac{dy}{2y}=\frac{dt}{3t}=-\frac{3du}{2u},$$

is

(3.15) $$r=xt^{-\frac{1}{3}},\ \ \ s=yt^{-\frac{2}{3}},\ \ \ u=t^{-\frac{2}{9}}W(r,s).$$

Substituting from (3.15) into (1.4) the latter reduces to

(3.16) $$(9W^3W_r\ -3rW_r\ -2W\ -6sW_s+9\alpha W_{rrr})_r\ +9\beta W_{ss}\ \ =0.$$

4. Reductions of (1.4) by two-dimensional sub algebras

As reductions can be facilitated using two types of two-dimensional sub algebras, namely, Abelian sub algebras and solvable non-Abelian sub algebras, we consider them separately.

(4.1) Two-dimensional Abelian sub algebras

Case 1. Sub algebra $L_{A,1}\ =\{V_1,V_2\}$

From Table 1 we find that the generators V_1 and V_2 commute, that is, $[V_1,V_2]=0$. We can initiate the reduction procedure by taking V_1 or V_2. If we begin with V_2, then (1.4) is reduced to the PDE (3.7) . We now write below V_2^* which is V_2, but, expressed in terms of the similarity variables given in (3.2):

(4.1) $$V_2^* = 2\beta r \partial_s.$$

The associated characteristic equation is $\frac{dr}{0} = \frac{ds}{2\beta r} = \frac{dW}{0}$, whose solution is $\rho = r$ and $W = H(\rho)$. Consequently, (3.3) is replaced by an ODE

(4.2) $$2\rho H' = F(\rho).$$

Case 2. Sub algebra $L_{A,2} = \{V_1, V_3\}$

It follows from Table 1 that $[V_1,V_3]$=0. We shall begin with V_3 to transform (1.4) to (3.10). Then V_1 changes to $V_1^* = \partial_r$. Integration of the characteristic equations associated with V_1^* gives $W=H(\rho)$, $\rho=r$ which reduces (3.3) to $H'=G(r)$.

Case 3. Sub algebra $L_{A,3}=\{V_1,V_4\}$

Since $[V_1,V_4]$=0, we begin with V_1 and arrive at the PDE (3.3). We express V_4 in terms of the similarity variables defined in (3.6) as $V_4^* = \partial_r$. As a result (3.7) reduces to H''=0.

Case 4. Sub algebra $L_{A,4}=\{V_3,V_4\}$

We begin with V_3. In this case (1.4) is reduced to the PDE (3.10).

We express V_4 in terms of the similarity variables defined (3.12) as

(4.3) $$V_4^* = \partial_r.$$

The characteristic equation for V_4^* is

(4.4) $$\frac{dr}{1} = \frac{ds}{0} = \frac{dw}{0}.$$

Integrating (4.4) we obtain the transformation $W = H(\rho)$, $\rho = r$ which replaces (3.10) by

(4.5) $$H^3 H' + \alpha H''' = G(\rho),$$

where $G(\rho)$ is an arbitrary function.

4.2. A two-dimensional solvable Non-Abelian sub algebra

The sub algebra $L_{nA,1} = \{V_4, V_5\}$ has the property $[V_4,V_5]=V_4$. With V_4 we transform (1.4) to (3.13). We express V_5 in terms of the similarity variables defined (3.9) as

(4.6) $$V_5^* = \frac{1}{3} r\partial_r + \frac{2}{3} s\partial_s - \frac{2}{9} W\partial_w.$$

The characteristic equation for V_5^* is

(4.7) $$\frac{dr}{\frac{r}{3}} = \frac{ds}{\frac{s}{3}} = \frac{dW}{-\frac{w}{9}}.$$

Integration of (4.7) leads to new variables and $W = s^{-\frac{1}{3}} H(\rho)$, $\rho = rs^{-\frac{1}{2}}$, where $H(\rho)$ satisfies the ODE

(4.8) $$(36H^3H' + 36\alpha H''' + 16\beta\rho H + 9\beta\rho^2 H')' + 5\beta\rho H' = 0.$$

5. A (2 + 1)-dimensional VCKdV equation with damping

In this chapter we consider the following (2 + 1)-dimensional VCKdV equations with damping

(5.1) $$(\lambda u^m + u_t + u^3 u_x + \alpha u_{xxx})_x + \beta u_{yy} = 0, \quad \alpha,\ \beta,\ \lambda \in R^+,$$

(5.2) $$(t^\lambda u + u_t + u^3 u_x + \alpha u_{xxx})_x + \beta u_{yy} = 0, \quad \alpha,\ \beta,\ \lambda \in R^+,$$

(5.3) $$(t^\lambda u^m + u_t + u^3 u_x + \alpha u_{xxx})_x + \beta u_{yy} = 0, \alpha,\ \beta,\ \lambda \in R^+.$$

For (5.1), the infinitesimals ξ, η, τ, ϕ and the four generators corresponding to each of the constants $c_i, i = 1, 2, 3$ & 4 are

(5.4) $$\xi = c_3 + c_4 y \quad \eta = c_1 = 2c_4 t\beta, \quad \tau = c_2, \quad \emptyset = 0,$$

and

(5.5) $V_1 = \partial_y,\ V_2 = \partial_t,\ V_3 = \partial_x,\ V_4 = y\partial_x - 2t\beta\partial_y.$

These symmetry generators form a closed Lie algebra as is seen from the following commutator Table:

Table 2

$[V_i,V_j]$	V_1	V_2	V_3	V_4
V_1	0	0	0	V_3
V_2	0	0	0	$-2\beta V_1$
V_3	0	0	0	0
V_4	V_3	$2\beta V_1$	0	0

For both the equations (5.2) - (5.3), the form of the infinitesimals ξ, η, τ, ϕ are the same:

(5.6) $\xi = c_2 + c_3 y \quad \eta = c_1 - 2c_3 t\beta,\ \tau = 0,\ \emptyset = 0$.

Now we write down the three symmetry generators corresponding to each of the constants $c_i, i = 1,2,3$ involved in the infinitesimals, viz.,

(5.7) $V_1 = \partial_y , V_2 = \partial_x,\ V_3 = y\partial_x - 2t\beta\partial_y.$

These symmetry generators form a closed Lie algebra as is seen from the following commutator Table:

Table 3

$[V_i,V_j]$	V_1	V_2	V_3
V_1	0	0	V_2
V_2	0	0	0
V_3	V_2	0	0

The reductions of one-dimensional sub algebras of equations (5.1) and (5.2, 5.3) are given in Table 4 ,Table 5 and Table 6.

Table 4

V_i	*Reduction*
V_1	$(W_r + W^3 W_s + \lambda W^m + \alpha W_{sss})_s = 0.$
V_2	$(W^3 W_r + \lambda W^m + \alpha W_{rrr})_r + \beta W_{ss} = 0.$
V_3	$W_{ss} = 0.$
V_4	$(2\lambda r W^m + 2sW_s + 4\beta r^2 W^3 W_s + 2rW_r + W + 16\alpha\beta^3 r^4 W_{sss})_s = 0$

Table 5

V_i	*Reduction*
V_1	$(r^\lambda W + W_r + W^3 W_s + \alpha u_{sss})_s = 0.$
V_2	$W_{ss} = 0.$
V_3	$\left(2sW_s + 4\beta r^2 W^3 W_s + 2rW_r + \left(2r^{\lambda+1} + 1\right)W + 16\alpha\beta^3 r^4 W_{sss}\right)_s = 0$

Table 6

V_i	*Reduction*
V_1	$\left(r^{\lambda}W^{m}+W_{r}\ +W^{3}W_{s}+\alpha u_{sss}\right)_{s}\ =0.$
V_2	$W_{ss}\ =0.$
V_3	$\left(2sW_{s}+4\beta r^{2}W^{3}W_{s}+2rW_{r}\ +2r^{\lambda+1}W^{m}+W\ +16\alpha\beta^{3}r^{4}W_{sss}\right)_{s}=0$

The reductions of two-dimensional sub algebras of equations (5.1) and (5.2,5.3) are given in Table 7 and Table 8.

Table 7

Algebra	*Reduction*
$[V_1,V_2]=0$	$(H^3H'+\lambda H^m+\alpha H''')'=0.$
$[V_2,V_3]=0$	$H''=0.$
$[V_3,V_4]=0$	$H'=L(\rho).$

Table 8

Algebra	*Reduction*
$[V_1,V_2]=0$	$H'=G_1(\rho)$
$[V_2,V_3]$	$H'=G_2(\rho)$

6. Conclusions

- In this paper a (2+1)-dimensional KdV equation with variable coefficient $(u_t+\ u^3u_x+\alpha u_{xxx})_x+\beta u_{yy}\ =0,\alpha,\beta\in R^{+}$ is subjected to Lie's classical method
- Equation (1.4) admits a five-dimensional symmetry group.
- It is established that the symmetry generators form a closed Lie algebra.
- Classification of symmetry algebra of (1.4) into one- and two-dimensional subalgebras is carried out.
- Systematic reduction to (1+1)-dimensional PDE and then to ODEs are performed using one-dimensional and two-dimensional Abelian and solvable nonAbelian subalgebras.
- A solution of (1.4) containg two arbitrary functions of *t* is determined by reduction to a linear partial differential equation.

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