Some Results On Power Graph Of Finite Cyclic Group

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Abstract *- Let be a group. The power graph (G) of G is a graph with vertex set $V(P(G)) = G$ and two distinct vertices x and y are adjacent in $P(G)$ if and only if either $x^i = y$ or $y^j = x$, where $2 \leq i, j \leq n$. We discuss some graph theoretical properties of power graph for some classes of finite cyclic group. Further we discuss some domination parameters of power graph for some classes of finite cyclic group.*

Keywords - *power graph, pan-cyclic, Hamiltonian, domination, chordal graph, split graph, finite group.*

I. Introduction

The study of algebraic structures using the properties of graphs becomes an exciting research topic in the last twenty four years, leading to many fascinating results and questions. Investigation of algebraic properties of groups or rings using the associated graph becomes an exciting topic. Defined the undirected power graph (G) of a groupG. Actually the power graph (G) of G is the graph with vertex set $V(P(G)) = G$ and two distinct vertices $x, y \epsilon G$ are adjacent in $P(G)$ if and only if either $x^i = y$or $y^i = x$ where i and are j integers and $2 \leq i, j \leq n$.

In this paper, we discuss the following properties such as Hamiltonian, pancyclic, split, chordal and weakly perfect of the power graph $P(G)$ where G is a finite cyclic group of n where $n = pq$ with $p < q$ and are distinct primes.Throughoutthis paper, we use the following notations and definitions.

By a graph $\Gamma = (V, E)$, we mean an undirected graph Γ with vertex set V, edge set E and has no loops or multiple edges. The degreeG(v) of a vertex v in Γ is the number of edges incident to v and if the graph is understood, then we denote $deg_\Gamma(v)$ simply by $\deg(v)$. The order of Γ is defined as $|(\Gamma)|$ and its maximum and its minimum degrees will be denoted, respectively, by $\Delta(P(G))$ and $\delta(\Gamma)$. A subset $X$of the vertices of Γ is called an *independent set* if the induced sub graph$\langle X \rangle$ on X has no edges. The maximum size of an independent set in a graph Γ is called the *independence number* of $\Gamma$and denoted by $\beta_0(\Gamma)$.The *union* $\Gamma_1 \cup \Gamma_2$ of two graphs $\Gamma_1 = (v_1, E_1)$ and $\Gamma_2 = (v_2, E_2)$ is the graph with vertex set $V = v_1 \cup v_2$and edge set $E = e_1 \cup e_2$. The *join* $\Gamma_1 + \Gamma_2$ of $\Gamma = (v_1, E_1)$ is the $\Gamma = (v_2, E_2)$ is the graph with $V = v_1 \cup v_2$ and $E = e_1 \cup e_2$ together with edges joining all vertices in $v_1$with vertices in v_2. Let G be a group with identity e. The number of elements of a group is called its *order* and it is denoted by (g). The order of an element g in a group is the smallest positive integer n such that $g^n = e$. If no such integer exists, we say g has infinite order. The order of an element g is denoted (g).

II. Properties of the power graph of finite cyclic graph

First we show that the power graph (G) is Hamiltonian where G is a finite cyclic group of $n = pq$ with where and p<q are distinct primes. Note that, a path in Γ that contains every vertex of Γ is called a Hamiltonian path of Γ, while a cycle in Γ that contains every vertex of Γ is called a Hamiltonian cycle of Γ. A graph that contains a Hamiltonian cycle is itself called Hamiltonian.

Lemma 1

Let G be a finite cyclic group of $n = pq$ with $p < q$ and are distinct primes. Then (G) is a Hamiltonian graph.

Proof

Assume that $n = pq$ with $p < q$ and are distinct primes.

Let $S = \left\{p^i : 1 \leq i \leq \frac{n}{p} - 1\right\}$ and $T = \left\{q^i : 1 \leq i \leq \frac{n}{q} - 1\right\}$.
Then $\langle S \rangle$ and $\langle T \rangle$ are complete sub graphs in (G). Note that $|S| = q$ and $|T| = p$.

Also, $\langle (\mathbb{Z}_n) \cup \{0\} \rangle$ is also a complete sub graph in $P(G)$. Note that $|(\mathbb{Z}_n) \cup \{0\}| = \varphi(n) + 1$.

Let P_1 be a spanning path in $\langle S \rangle$ such as $p - p^3 - \cdots - p^{\bar{q}} - p^2$.

Let P_2 be a spanning path in $\langle T \rangle$ such as $q - q^3 - \cdots - q^{\bar{q}} - q^2$.

Let P_3 be a spanning path in $\langle U(\mathbb{Z}_n) \cup \{0\} \rangle$ such as $u_1 - u_2 - \cdots - u_{\varphi(n)}$. Thus the spanning cycle $0 - P_1 - P_3 - P_2 - 0$ in (G), where 0 is the identity in $(\mathbb{Z}_n, \oplus_n)$.

By definition of a Hamiltonian graph, (G) is a Hamiltonian graph.

In following lemma, we discuss the pancyclic nature of the power graph.

Note that, A graph Γ of order $m \geq 3$ is pancyclicif Γ contains cycles of all lengths from 3 to m. Also Γ is called vertex-pancyclic if each vertex m of Γ belongs to a cycle of every length l for $3 \leq l \leq m$.

Lemma 2

Let G be a finite cyclic group of on n $= pq$ with $p < q$ and are distinct primes.

Then (G) is pancyclic. Moreover, (G) is vertex-pancyclic.

Proof

Consider P_1, P_2 andP_3 as specified in Lemma 1.

Consider the spanning cycle $0 - P_1 - P_3 - P_2 - 0$ in (G). Now removing vertex one by one from P_1 and after that P_2, finally proceeding this process for P_3. This gives that (G) contains a cylcle of length m, where $3 \leq m \leq |(P(G))|$. Thus (G) is pancyclic.

In view of the above argument, it is easy to see that (G) is vertex-pancyclic.
In following lemma, we discuss the split nature of the power graph.

Note that, a split graph is a graph in which the vertices can be partitioned into the disjoint union of an independent set and a clique (either of which may be empty).

Lemma 3

Let G be a finite cyclic group of n $= pq$ with $p < q$ and are distinct primes. Then $(G)$is a split graph if and only if

$p = 2$.

Proof

Assume that $n = pq$ with $p < q$ and $p = 2$. Consider $S = \mathbb{Z}_n \backslash \{q\}$ and $T = \{q\}$.

Note that $\langle S \rangle$ *and* $\langle T \rangle$ are complete sub graphs in (G). Also, $|S| = n - 1$ and $|T| = 1$. Consider $\langle S \rangle$ is a complete sub graph in (G) and $\langle T \rangle$ is an independent sub graph in(G). By the definition of split graph, (G) is a split graph.

In following lemma, we discuss the chordal nature of the power graph.

Recall that, a chordal graph is a simple graph in which every graph cycle of length four and greater has a cycle chord.

Lemma 4

Let G be a finite cyclic group of n= pq with $p < q$ and are distinct primes. Then (G) is a chordal graph.

Proof

Note that $(G) = (K_{p-1} \cup K_{q-1}) + {}_{(pq)+1}$. Consider S *and* T as specified in

Lemma 1.

Consider $\langle S \rangle = K_{q-1}$and $\langle T \rangle = K_1$. Also, $\langle (\mathbb{Z}_n) \cup \{0\} \rangle = K_{\varphi(pq)+1}$.

Suppose that $C_m$is an induced cylcle sub graph in (G) where $m \geq 4$.

Suppose that $C_m$contains one vertex V from $\langle (\mathbb{Z}_n) \cup \{0\} \rangle$. Then G$(v) \geq 3$.

It is impossible because degree of any vertex in is 2. This turns that contains no vertex from $\langle U(\mathbb{Z}_n) \cup \{0\} \rangle$. Suppose contains at least one vertex from $\langle S \rangle$ and at least one vertex from $\langle T \rangle$.

Then C_m contains two connected components since no vertex from $\langle S \rangle$ is adjacent to any vertex in $\langle T \rangle$. This implies that C_m is disconnected, which is contradiction to C_m is connected. Hence C_m is a sub graph of either $\langle S \rangle$ or $\langle T \rangle$. Without loss of generality assume that C_m is a sub graph of $\langle S \rangle$. Then C_m is a complete sub graph in $\langle S \rangle$. Then C_m is a complete sub graph in $\langle S \rangle$ as well as in (G), which is contradiction to is an induced cycle sub graph in (G). Thus (G) contains no induced cylcle sub graph of order at least $m \geq 4$. Hence (G) is a chordal graph.

In following lemma, we obtain the clique number of the power graph.

Lemma 5

Let G be a finite cyclic group of n= pq with $p < q$ and are distinct primes. Then $((G)) = n - p$.

Proof

Consider and T as specified in Lemma 1.

Note that $\langle T \rangle$ is complete sub graph in (G).Also $|S| = q$ and $|T| = p$.

Suppose $\langle(\mathbb{Z}_n\backslash T)\cup\{x\}\rangle$ is not a complete sub graph in $P(G)$ where $x \in T$. This implies that $\langle\mathbb{Z}_n\backslash T\rangle$ is the maximal complete sub graph in (G). Hence $(\ (G)) = n - p$.

In following lemma, we obtain the chromatic number of the power graph.

Lemma 6

Let G be a finite cyclic group of n= pq with $p < q$ and are distinct primes. Then $(\ (G)) = n - p$.

Proof

As observed in pervious lemma, $\langle T\rangle$ is the maximal complete sub graph in (G).

Also $S \subseteq T$ and $|S| > |T|$. We know that no vertex in $\langle S\rangle$ is adjacent to any vertex from $\langle T\rangle$ in (G). Note that $(G) = \langle T_1 = \mathbb{Z}_n\backslash T\rangle\cup\langle T\rangle$. Since $\langle T\rangle$ is a complete sub graph in (G),$|\langle T\rangle| = p$. Since $|S| > |T|$, let P colors which is used in $\langle S\rangle$ assign to the P vertices in $\langle T\rangle$. This implies that (G) exactly needs $n - p$ colors.

Hence $(\ (G)) = n - p$.

In following lemma, we discuss the weakly perfect nature of the power graph.

Note that, a graph Γ is said to be weakly perfect if (Γ) = (Γ).

Example:

Let G be a cyclic group of order 15

Solution:

Consider S ={0,3,6,9,12} and T ={5,10}

Consider 0-3-6-9-12-1-5-10-2-4-7-8-11-13-14-0 is a spanning cycle in G. (using lemma 2)

Removing vertex one by one from S∪T∪V(G) then (G) has a cycle of length m. where 1≤ m≤ n .Hence (G) is both pancyclic and vertex pancyclic.

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