Some Properties Of Simple Ternary Semigroups

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***ABSTARCT**: Completely simple semigroups was studied by P. R. Jones and simple ternary semigroups was studied by G. Sheeja and S. Sri Balaji in 2013. In this paper we considered Simple Ternary semigroups and we proved some proved their some properties. Mainly we prove that If $(T,.)$ is a ternary semi-group then T is left, right and lateral simple if and only if every fuzzy quasi ideal of T is a constant function.*

***Key Words:** Ternary Semigroup, Regular, Simple Semigroup, Ideals, Fuzzy Quasi ideal.*

1. INTRODUCTION:

A semi-group not containing proper ideals or congruences of some fixed type. Various kinds of simple semi-groups arise, depending on the type considered: ideal-simple semi-groups, not containing proper two-sided ideals (the term simple semi-group is often used for such semi-groups only); left (right) simple semi-groups, not containing proper left (right) ideals; (left, right) 0-simple semi-groups, semi-groups with a zero not containing proper non-zero two-sided (left, right) ideals and not being two-element semi-groups with zero multiplication; bi-simple semi-groups, consisting of one $\mathcal{D}$-class (cf. Green equivalence relations); 0-bi-simple semi-groups, consisting of two $\mathcal{D}$-classes one of which is the null class; and congruence-free semi-groups, not having congruences other than the universal relation and the equality relation. In 1932, Lehmer introduced the concept of a ternary semigroup. He investigated certain ternary algebraic structures called triplexes. Santiago developed the theory of ternary semigroups and semiheaps. He studied regular and completely regular ternary semigroups. The notion of quasi ideals and bi-ideals in ternary semigroups presented by Dixit and Dewan, Kar. Maity investigated congruences of ternary semigroups and Iampan studied minimal and maximal lateral ideals of ternary semigroups. In this paper, we proved some results on simple ternary semigroups.

Definition 1.1: A non-empty set T is said to be ternary semigroup if there exists a ternary operation $\cdot : T \times T \times T \to T$ written as $(a,b,c) \to a.b.c$ satisfies the following identity $ab(cde) = a(bcd)e = (abc)de$ for any $a,b,c,d,e \in T$.

Definition 1.2: A ternary semigroup T is said to be a ordered ternary semigroup if T is a ordered set with the relation "$\le$" such that $a \le b \Rightarrow aa_1a_2 \le ba_1a_2$, $a_1aa_2 \le a_1ba_2$ and $a_1a_2a \le a_1a_2b$ for all $a,b,a_1,a_2 \in T$.

Definition 1.3: A non-empty subset A of a ordered ternary semigroup T is called a Left (lateral, right) ideal, of T if it satisfies following:

1) $TTA \subseteq A (ATT \subseteq A, TAT \subseteq A\ respectively)$
2) If $a \in A$ & $b \in T$ such that $b \le a$ then $b \in A.$

Definition 1.4: Let X is a non empty set. A fuzzy set μ of the set X is a function $\mu : X \to [0,1].$

Definition 1.5: Let $F(T)$ denote the set of all fuzzy sets in a ternary semigroup T . For A,B,C$\in F(T)$, $A \subseteq B$ and $B \subseteq C$ if and only if $A(x) \le B(x)\ and\ B(x) \le C(x)$ in the ordering of [0,1], $\forall x \in T.$

Definition 1.6: A fuzzy set $A \in F(T)$ is said to be a fuzzy sub semigroup of a ternary semigroup T if $A(xyz) \ge \min\{A(x), A(y), A(z)\} \forall x, y, z \in T.$

Definition 1.7: A fuzzy set $A \in F(T)$ is said to be a fuzzy left (resp., lateral and right) ideal of a ternary semigroup T if $A(xyz) \ge A(z),$ (resp., $A(xyz) \ge A(x),$ and $A(xyz) \ge A(y)$) $\forall x, y, z \in T$.

Definition 1.8: Let f and g be two fuzzy subset of T , define the relation $\subseteq$ between f and g respectively as,

$f \subseteq g$ if $f(x) \le g(x)$
$(f \cup g)(x) = \max\{f(x), f(y)\}$
$(f \cap g)(x) = \min\{f(x), g(x)\}, \forall x \in T$

Definition 1.9: A ternary semigroup T is called left (right/lateral) simple if for every left (right/lateral) ideal A of T, we have $A = T$.

Definition 1.10: An ternary semigroups (T ,.) is called left (right, lateral) regular. If for each $a \in T$ there exits $x, y \in T$ such that $a = a^3xy (a = xya^3, a = xa^3y)$. An ternary semigroup ($T$,.) is called strongly regular if it is left, right & lateral regular.

Clearly, an ternary semigroup T is strongly regular if and only if `$a \in TTa^3 \bigcap a^3TT \bigcap Ta^3T$ for every $a \in T$.

Definition 1.11: A ternary semigroup (T ,.) is called regular if for each $a \in T$ there exist $x \in T$ such that $a = axaxa$.. Equivalently, $a \in aTaTa$ for any $a \in T$.

Definition 1.12: An equivalence relation ρ on an ordered ternary semigroup T is called congruence if $(a,b,c) \in \rho$ implies $(acd, bcd) \in \rho, (cad, cbd) \in \rho$ for every $c, d \in T$.

Definition 1.13: A congruence ρ on T is called semi lattice congruence on T ,if $(a, a^3) \in \rho$

Definition 1.14: A ternary semigroup T is called a semi lattice of left, right & lateral simple semigroup if there exists a semi lattice congruence ρ on T such that the $\rho -$class $(x)\rho$ of T containing x is a left, right & lateral simple sub semi group of T for every $x \in T$.

Equivalently, there exists a semi lattice Y and a family $\{T_\alpha\}_{\alpha \in Y}$ of left, right and lateral simple sub semigroups of T such that

i. $T_\alpha \cap T_\beta \cap T_\gamma = \phi$ for each $\alpha, \beta, \gamma \in Y, \alpha \ne \beta, \beta \ne \gamma$;
ii. $T = U_{\alpha \in Y} T_\alpha$;
iii. $T_\alpha T_\beta T_\gamma \subseteq T_{\alpha\beta\gamma}$ for each $\alpha, \beta, \gamma \in Y$.

Definition 1.15: A ternary sub semigroup F of an ordered ternary semigroup T is called a filter of T if

1. $a, b, c \in T, abc \in F \Rightarrow a \in F, b \in F \; \& \; c \in F$;
2. $a \in F, T \ni c \ge a \Rightarrow c \in F$.

We denote by $N(a)$ the filter of T generated by $a (a \in T)$ and by "N" the equivalence relation on T defined by

$aNb \Leftrightarrow N(a) = N(b)$.

N is semi-lattice congruence on T .

2. Maine Results:

Lemma 2.1: Let T be an ordered ternary semigroup. Then T is Left (Right/Lateral) simple if and only if $TTa = T (aTT = T \;\; or \;\; TaT = T)$ for every $a \in T$.

Lemma 2.2: Let T be an ordered semigroup. If T is Left, Right and Lateral simple, then T is regular.

Proof: Let $a \in T$

By hypothesis, $T = aTT = TTa = TaT$

Then we have, $a \in aTT = aTTaTTa$
$= aTaTa$
$\therefore a = aTaTa$

It follows that T is regular.

Theorem 2.3: Let (T ,.) be a ordered ternary semigroup .Then T is left, right and lateral simple if and only if every fuzzy quasi ideal of T is a constant function.

Proof: Suppose that T is left, right and lateral simple ternary semigroup.

Let f be a fuzzy quasi ideal of T and $a \in T$.

We consider the set, $E_T = \{e \in T / e^3 = e\}$

Then E_T is non-empty.

There exist $x \in S$ such that $a = axaxa$. then we have,

$(ax)^3 = axaxax$

$= axaxa$

$= ax$

So, $ax \in E_T$.

1) f is a constant mapping on E_T.

Let $t \in E_T$. Then $f(e) = f(t)$ for every $e \in E_T$.

In fact, Since T is left, right and lateral simple .We have,

$TTt = T, tTT = T \text{ and } TtT = T$

Since $e \in T$, we have, $e \in TTt, e \in tTT \text{ and } e \in TtT$

So, there exist $x, y \in T$ such that $e = xyt, e = txy \text{ and } e = xty$.

Hence,

$e^3 = eee = (xyt)(xyt)(xyt) = (xytxytx)yt$

$e^3 = eee = (txy)(txy)(txy) = t(xytxytx)y$

$e^3 = eee = (xty)(xty)(xty) = xt(yxtyxty)$

And we have,

$((xytxytx), y, t) \in A_{e^3}, (t, (xytxytx), y) \in A_{e^3} \text{ and } (x, t, (yxtyxty)) \in A_{e^3}$.

Since f is a fuzzy quasi ideal of T,

we have

$$f(e^3) = ((f \circ T \circ T) \cap (T \circ T \circ f) \cap (T \circ f \circ T))(e^3)$$

$$= \min\{(f \circ T \circ T)(e^3), (T \circ T \circ f)(e^3), (T \circ f \circ T)(e^3)\}$$

$$= \min\{\bigvee_{p,q,r \in A_{e^3}} \min\{f(p), T(q), T(r)\}, \bigvee_{u,v,w \in A_{e^3}} \min\{T(u), T(v), f(w)\},$$
$$\bigvee_{i,j,k \in A_{e^3}} \min\{T(i), f(j), T(k)\}\}$$

$$= \min\{\min\{f(t), S(xytxytx), S(y)\}, \min\{S(xytxytx), S(y), f(t)\},$$
$$\min\{S(x), f(t), S(yxtyxty)\}\}$$

$$= \min\{\min\{f(t),1,1\}, \min\{1,1, f(t)\}, \min\{1, f(t,)\}\}$$

$$= f(t).$$

Since $e \in E_T$, it follows that $e^3 = e$ and f is a fuzzy quasi ideal of T, and we have $f(e) = f(e^3)$.

Thus, $f(e) = f(t)$.

On other hand, Since T is left, right and lateral simple and $e \in T$, we have

$TTt = T, tTT = T \text{ and } TtT = T$

Since, $t \in E_T \subseteq T$, as in the previous case we also have

$f(t) \geq f(t^3) \geq f(e)$.

2) f is a constant mapping on T.

Let $a \in T$ then $f(t) = f(a)$ for every $t \in E_T$.

Since T is regular, there exist $x \in T$ such that $a = axaxa$.

Then,

$(ax)^3 = (axax)ax$

$= a(xaxa)x$

$= ax(axax)$.

This implies that, $aax, xaa, axa \in E_s$

Then by (1), we have $f(aax) = f(t), f(xaa) = f(t)$ and $f(t)$.

$(aax)(axaxa)(axaxa) \geq axaxa \geq a$

Since, $(axaxa)(axaxa)(xaa) \geq axaxa \geq a$ and

$(axaxa)(axa)(axaxa) \geq axaxa \geq a$

We have $(aax, axaxa, axaxa) \in A_a$ & $(axaxa, axaxa, xaa) \in A_a$ and $(aax, axaxa, axaxa) \in A_a$.

Since f is a fuzzy quasi-ideal of s, we have

$$f(a) = ((f \circ T \circ T) \cap (T \circ T \circ f) \cap (T \circ f \circ T))(a)$$
$$= \min\{(f \circ T \circ T)(a), (T \circ T \circ f)(a), (T \circ f \circ T)(a)\}$$
$$= \min\{ \bigvee_{p,q,r \in A_a} \min\{f(p), T(q), T(r)\}, \bigvee_{u,v,w \in A_a} \min\{T(u), T(v), f(w)\}, \bigvee_{i,j,k \in A_a} \min\{T(i), f(j), T(w)\}\}$$
$$= \min\{\min\{f(aax), T(axaxa), T(axaxa)\}, \min\{T(axaxa), T(axaxa), f(xaa)\}, \min\{T(axaxa), f(axa), T(axaxa)\}$$
$$= \min\{\min\{f(t),1,1\}, \min\{1,1, f(t)\}, \min\{1, f(t),1\}$$
$$= f(t).$$

On the other hand, since T is left, right, &lateral simple, we have

$TTa = T, aTT = T$ & $TaT = T, t \in TTa, t \in aTT$ & $t \in TaT$.

Then $t \leq uua, t \leq avv$ & $t \leq waw$ for some $u, v, w \in T$.

Then $(u,u,a) \in A_t, (a,v,v) \in A_t$ & $(w,a,w) \in A_t$.

Since f is a fuzzy quasi-ideal of T , we have

$$f(t) = ((f \circ T \circ T)(t), (T \circ T \circ f)(t), (T \circ f \circ T)(t)\}$$
$$= \min\{ \bigvee_{x_1, y_1, z_1 \in A_t} \min\{f(x_1), T(y_1), T(z_1)\} \bigvee_{x_2, y_2, z_2 \in A_t} \min\{T(x_2), T(y_2), f(z_2)\} \bigvee_{x_3, y_3, z_3 \in A_t} \min\{t(x_3), f(y_3), T(z_3)\}\}$$
$$= \min\{\min\{f(a), T(v), T(v)\} \min\{T(u), T(u), f(a)\} \min\{T(w), f(a), T(w)\}$$
$$= \min\{\min\{f(a),1,1\} \min\{1,1, f(a)\} \min\{1, f(a),1\}$$
$$= f(a).$$

Summarizing two cases above, we have shown that f is a constant function.

Conversely,

Let $a \in T$ since the set TTa is a left ideal of T , and so TTa is a quasi-ideal of T .

The characteristic function f_{TTa} of Tta is fuzzy quasi-ideal of T . By hypothesis, f_{TTa} is a constant function, that is there exist $c \in \{0,1\}$ such that $f_{TTa}(x) = c$ for every $x \in T$ .Let $aTT \subset T$ and t be an element of T such that $t \notin aTT$ .Then $f_{(aTT)}(t) = 0$.

Also since $a^3 \in TTa, f_{TTa}(a^3) = 1$, leading to a contradiction to the fact that f_{TTa} is a constant function thus $TTa = T$

Similarly, we can prove that $aT = T$ & $TaT = T$

Therefore T is left, right & lateral simple

Lemma 2.4: Let T be a ternary semigroup. Then the following statements are equivalent: $(x)NN$ is a left, right & lateral simple sub semi group of T for every $x \in T$

Lemma 2.5: A ternary semigroup $(T,.)$ is a semi lattice of left, right and lateral simple semigroup if and only if $(A) = A$ and

$ABC = BCA = CBA = CAB = BAC = ACB$ for all quasi-ideals ABC of T .

Theorem 2.6: Let $(T,.)$ be a ternary semigroup. Then S is a semi lattice of left, right and lateral simple semigroups if only if for every fuzzy quasi-ideal f of T, we have $f(a)=f(a^3)$ and $f(abc)=f(cba)=f(cab)=f(bca)=f(bac)=f(acb)$ for all $a,b,c\in T$

Proof: Let Y be a semi lattice and let $\{T_\alpha\}_{\alpha\in Y}$ be a family of left and right simple sub semigroups of ternary semigroup T such that

i. $T_\alpha\cap T_\beta\cap T_\gamma=\phi$ for each $\alpha,\beta,\gamma\in Y, \alpha\neq\beta, \beta\neq\gamma$.

ii. $T=U_{\alpha\in Y}T_{\alpha i}$

iii. $T_\alpha T_\beta T_\gamma\subseteq T_{\alpha\beta\gamma}$ for each $\alpha,\beta,\gamma\in Y$

Let f be a fuzzy quasi-ideal of T .Then we have

i. Let $a\in T$. Then $f(a)=f(a^3)$ indeed

It is sufficient to prove that T is strongly regular,

Since $a\in T=U_{\alpha\in Y}T_\alpha, \exists\alpha\in Y$ such that $a\in T_\alpha$. since T_α is left, right and lateral simple, we have

$T_\alpha=T_\alpha T_\alpha a$

$T_\alpha=aT_\alpha T_\alpha$

$T_\alpha=T_\alpha aT_\alpha$

Then we have $(aT_\alpha T_\alpha)=(aT_\alpha T_\alpha aT_\alpha T_\alpha a)$

$=(aT_\alpha aT_\alpha a)$

Since $a\in\alpha$, we have $a\in(aT_\alpha aT_\alpha a)$, then there exists $x\in T_\alpha$ such that $a\le axaxa$. since $x\in(aT_\alpha aT_\alpha a)$

There exists $y\in T_\alpha$ such that $x\le ayaya$.

Thus, $a\le axaxa$

$\le a(apa)a(apa)a$

$\le a(apapa)a(apapa)a$

$\le aa(aqaqa)apaaa(apaqaqaa)a$

$\le aaa(qaaap)aaa(paqaq)a^3$

$\le a^3(qa^3p)a^3(paqaq)a^3$

$\subseteq a^3T_\alpha a^3T_\alpha a^3$

$\subseteq(a^3T_\alpha T_\alpha)\cap(T_\alpha a^3T_\alpha)\cap(T_\alpha T_\alpha a^3)$

$\Rightarrow T$ is strongly regular.

ii. Let $a,b,c\in T$.

Then $f(abc)=f(bca)=f(cab)=f(cba)=f(bac)=f(acb)=f(bca)$

$(abc)^5=abcabcabcabcabc$

$=Q(abcab)Q(cabca)Q(bcabc)$

$=Q(bcabc)Q(abcab)Q(cabca)$

$=[bcabc\cup(bcabcTT\cap TbcabcT\cap TTbcabc)]$

$[abcab\cup(abcabTT\cap TabcabT\cap TTabcab)]$

$[cabca\cup(cabcaTT\cap TcabcaT\cap TTcabca)]$

$\subseteq[bcabc\cup(bcabcTT][abcab\cup TabcabT)][cabca\cup TTcabca]$

$\subseteq(bcabcTt)(TabcabT)(TTcabca)$

$(\because bcT\subset T, Tab\subset T, Tca\subset T)$

$$\subseteq [bcaTt][TcabT][TTbca] \qquad \because RML = R \cap M \cap L$$
$$= (bcaTt) \cap (TcabT) \cap (TTbca)$$
$$\because (abc)^5 \in (bcaTT) \cap (TcabT) \cap (TTbca)$$

$\because f$ is quasi-ideal.

$(f \circ T \circ T) \cap (T \circ f \circ T) \cap (T \circ T \circ f) \subseteq f$ &

$(f \circ T \circ T) \cap (T \circ T \circ f \circ T \circ T) \cap (T \circ T \circ f) \subseteq f$

$[(f \circ T \circ T) \cap (T \circ f \circ T) \cap (T \circ T \circ f)](abc)^5 \subseteq f(abc)^5$

$$f(abc)^5 \geq \min\{(f \circ T \circ T)(abc)^5, (T \circ f \circ T)(abc)^5, (T \circ T \circ f)(abc)^5\}$$
$$= \min\{ \bigvee_{p,q,r \in A_{(abc)^5}} \min\{f(p), T(q), T(r)\}, \bigvee_{t,s,u \in A_{(abc)^5}} \min\{T(t), f(s), T(u)\}, \bigvee_{x,y,z \in A_{(abc)^5}} \min\{T(x), T(y), f(z)\}\}$$
$$= f(abc)$$
$$\therefore f((abc)^5) = f(abc).$$

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