Proposed Piecewise Linear Approximation For Solving Nonlinear Transportation Problems

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ABSTRACT

The study has developed an algorithm that can be employed to tackle optimization problems with nonlinear transportation problem. The algorithm used two breakpoints. After transforming the nonlinear objective function to linear function, Wolfram Mathematica was employed to get the optimal solution. The two problems solved in this study showed that the proposed algorithm converged faster than solving the original problem directly via Wolfram Mathematica, though with the same optimal solution. In the study, a program code via Wolfram Mathematica for evaluating a nonlinear transportation problem with multiple linear variables and constraints of different sizes up to 70,000 variables and 35,000 constraints was written using the proposed technique. The study demonstrated the effectiveness of the proposed approach using the written program code as compared to the written program code for the original problem as it converged faster via the time of execution.

Keywords: *Transportation Algorithm, Nonlinear Transportation Problem, Piecewise Linear Approximation, Total Minimum Transportation Cost.*

INTRODUCTION

Volume discount plays a vital role in a transportation problem. The costs of goods are determined by factors such as: the costs of raw materials, labour, and transport. When cost of raw materials increases, so does the cost of the goods. Transportation cost also affects the pricing system. It is assumed that the cost of goods per unit shipped from a given source to a given destination is fixed regardless of the volume shipped. But in actuality the cost may not be fixed. Volume discounts are sometimes available for large shipments so that the marginal costs of shipping one unit might follow a particular pattern. Hence, the focus will be to develop a mathematical model using optimization techniques to close the demand and supply gap by discounting so as to minimize total transportation cost (Abdul-Salam, 2014).

The transportation problem which, being an aspect of network programming problems is a problem that deals with distributing any commodity from any group of 'sources' to any group of destinations or 'sinks' in the most cost effective way with a given 'supply' and 'demand' constraints . Depending on the nature of the cost function, the transportation problem can be categorized into linear and nonlinear transportation problem. In the linear transportation problem (ordinary transportation problem) the cost per unit commodity shipped from a given source to a given destination is constant, regardless of the amount shipped. It is always supposed that the mileage (distance) from every source to every destination is fixed. To solve such a transportation problem, the streamlined simplex algorithm is very efficient. However, in reality, there are at least two cases that the transportation problem fails to be linear (Ekezie and Opara, 2013).

First, the cost per unit commodity transported may not be fixed for volume discounts sometimes are available for large shipments. This would make the cost function either piecewise linear or just separable concave function. In this case the problem may be formulated as piecewise linear or concave programming problem with linear constraints. Second, in special conditions such as transporting emergency materials when natural calamity occurs or transporting military during war time, where carrying network may be destroyed, mileage from some sources to some destination are no longer definite. So the choice of different measures of distance leads to nonlinear (quadratic, convex …) objective function. In both cases, solving the transportation problem is not as simple as that of the linear one. In this study, a new algorithm was proposed to tackle nonlinear transportation problem. In particular, the nonlinear transportation problem considered in this study is stated as follows;

- We are given a set of n sources of commodity with known supply capacity and a set of m destinations with known demands.
- The function of transportation cost, nonlinear, and differentiable for a unit of product from each source to each destination.
- We are required to find the amount of product to be supplied from each source (may be market) to meet the demand of each destination in such a way as to minimize the total transportation cost.

The approach to solve this problem is applying the proposed technique.

Review of Related Literatures

Osuji et al (2014) worked on a research titled "application of transportation algorithm with volume discount on distribution cost using Nigerian Bottling Company Plc Owerri Plant". The study intended to obtain the quantity of Fanta (in crates), Coke (in crates) and Sprite (also in crates) that the Company should distribute in a month in order to minimize transportation cost and maximize profit. The authors referred the problem as a Nonlinear Transportation Problem (NTP), which was formulated in mathematical terms and tackled by the Karush-Kuhn-Tucker (KKT) optimality condition for the NTP. TORA software package was employed to determine the initial basic feasible solution via the Least Cost Method. The result of the analysis showed that the optimal solution that produced minimum achievable cost of supply was the supply of 5000 crates of Fanta and 6000 crates of the same product to Umuahia market zone and Afikpo respectively. 7000 crates of Coke, 9000 crates and 1000 crates of the same product should be supplied to Orlu, Mbaise, and Afikpo market zones respectively. 6000, and 5000 crates of Sprite should be allocated to Mbaise and Umuahia market zones respectively, at a total cost of ₦377, 000.

A study was carried out by Opara et al (2015) on nonlinear Transportation problems Algorithms using Nigerian Bottling Company Ltd Owerri Plant. The study intended to determine the quantity of Fanta, Coca-Cola, Schweppes and Sprite (all in crates) that the Company should distribute in a month in order to minimize the cost of transportation and maximize profit. Hence; the authors identified the problem as a Nonlinear Transportation Problem (NTP), formulated in mathematical models and tackled by the Karush-Kuhn-Tucker (KKT) optimality condition for the NTP. The initial basic feasible solution using the Least Cost Method was obtained via TORA software package. Thus, the result of the analysis revealed that the optimal solution that produced minimum achievable cost of supply was the supply of 8000 crates of Sprite and 5000 crates of the same product to Umuahia market zone and Eket respectively; 12000 crates of Coca-cola, and 4000 crates of the same product to Nnewi and Eket respectively; 9000 crates and 2000 crates of Schweppes product to Aba, and Umuahia market zones respectively; 1000 and 14000 crates of Fanta to Aba and Orlu market zones respectively, at a total cost of ₦509, 000.

From the literatures reviewed in this study, it has been observed that only one technique known as "Karush-Kuhn-Tucker (KKT) optimality condition for the NTP" was employed for the problem. Hence, this study proposed a new technique "piecewise linear approximation for solving nonlinear transportation problems"

DESCRIPTION OF THE PROPOSED APPROACH

In describing the proposed approach, consider the non-linear transportation problem of the form:

$$\left.\begin{aligned} \min .z &= \sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij}x_{ij} + p_{ij}x_{ij}^{2}\right) \\ s.t.: \sum_{j=1}^{n} x_{ij} &\le a_i, i = 1,2,\ldots,m \\ \sum_{i=1}^{m} x_{ij} &\ge b_j, j = 1,\ldots,n \\ x_{ij} &\ge 0, \forall i, j \end{aligned}\right\} \quad \ldots \quad (1)$$

Where $z(x)$ is a nonlinear separable function such that;

$$z(x)=z_{11}(x_{11})+z_{12}(x_{12})+...+z_{1n}(x_{1n})+z_{21}(x_{21})+z_{22}(x_{22})+...+z_{2n}(x_{2n})+...+z_{m1}(x_{m1})+z_{m2}(x_{m2})+...+z_{mn}(x_{mn})$$

$$\therefore z(x)=\sum_{i=1}^{m}\sum_{j=1}^{n} z_{ij}(x_{ij}) \qquad \ldots \qquad (2)$$

Here we say that a function $z(x_{11},\ldots,x_{1n},x_{21},\ldots,x_{2n},\ldots,x_{m1},\ldots,x_{mn})$ is separable if it can be expressed as the sum of mn single-variable functions as stated in Equation (2).

The first step of this algorithm is to introduce the Taylor series expansion as:

$$e^{x_i}=1+x_i+\frac{x_i^2}{2!}+\frac{x_i^3}{3!}+...=\sum_{k=0}^{\infty}\frac{x_i^k}{k!} \qquad \ldots \qquad (3)$$

Hence Equation (3) is stated as shown in Equation (4)

$$e^{x_i}=\sum_{k=0}^{\infty}\frac{x_i^k}{k!}, i=1,2,...,m \qquad \ldots \qquad (4)$$

Making x_i^2 the subject of the relation in Equation (3), we get Equation (5)

$$x_i^2=2\left[e^{x_i}-1-x_i-\sum_{k=3}^{\infty}\frac{x_i^k}{k!}, i=1,2,..,m\right] \qquad \ldots \qquad (5)$$

For this algorithm, we shall employ dichotomous breakpoints (1/2 & 1) and evaluate x_i^2 in Equation (5) given $x_i=\frac{1}{2}$ *and* 1. The values obtained shall be used to form a linear function which approximates the nonlinear objective function. Evaluation of x_i^2 (for i = 1,2,…,m) in Equation (5) gave 0.25 and 1 respectively, which then forms $0.25x_{i1}+x_{i2}$ (for i = 1,2,…,m), and the result will be substituted into the objective function of Equation (1) to give Equation (6).

Employing Equations (3) to (5) and substituting the result in the objective function of Equation (1) and modifying the constraints to give Equation (6).

$$\left.\begin{aligned} &\min .z=\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij}x_{ij}+p_{ij}(0.25x_{i1}+x_{i2})\right)\\ &s.t.:\sum_{j=1}^{n}x_{ij}\le a_i, i=1,2,...,m\\ &\sum_{i=1}^{m}x_{ij}\ge b_j, j=1,...,n\\ &x_{i1}+x_{i2}-x_i=0, i=1,2,...,m\\ &0\le x_{ij}\le a_i', \forall i,j \end{aligned}\right\} \qquad \ldots \qquad (6)$$

Where a_i' is the highest value amongst the $a_{i's}$.

Computer Written Program for Multiple Variables and Constraints for Nonlinear Transportation Problem

```
Clear[m,n,r,g,f,k,s,h,w,v,x,tim];
m=p (Number of sources);
n=q (Number of destinations);
r=Sum[-x[n*i-c]^2+x[n*i-c],{c,0,n-1},{i,1,n}];
f=(r+g);
k=Table[Sum[x[n*i-c],{c,0,n-1}]<=i+n,{i,1,n}];
s=Table[Sum[x[m*i+c],{i,0,m-1}]>=c+m,{c,1,m}];
h=Table[x[i]>=0,{i,1,n*(m)}];
```

```
w=Join[k,s,h];
v=Table[x[i],{i,1,n*(m)}];
tim=FindMinimum[{r,w},v]
```

Computer Written Program for Multiple Variables and Constraints for Nonlinear Problems for the Proposed Algorithm

```
Clear[m,n,r,g,f,f1,f2,k,s,h,w,v,c,x,p,u,t,i,z]
m= p(Number of sources);
n=q(Number of destinations);
r=m*n;
c=2n;
f1=Sum[x[n*i-c],{c,0,n-1},{i,1,n}];
f2=Sum[(-0.25*x[2i-1+r]-0.5*x[2i+r]),{i,1,r}];
f=(f1+f2);
u=Variables[f];
z=Length[u];
k=Table[Sum[x[n*i-c],{c,0,n-1}]<=i+n,{i,1,n}];
s=Table[Sum[x[m*i+c],{i,0,m-1}]>=c+m,{c,1,m}];
p=Table[x[2i+r-1]+x[2i+r]-x[i]==0,{i,1,r}];
h=Table[x[i]>=0,{i,r+1,z}];
v=Table[0<=x[i]<=c,{i,1,r}];
w=Join[k,s,p,h,v];
tim=NMinimize[{f,w},u]
```

4-dimensional example

Solve the Nonlinear programming problem (NLPP) using the proposed algorithm.

$$\min .z = -2x_{11}^2 - 5x_{12}^2 - x_{21}^2 - 2x_{22}^2 + 4x_{11} + 3x_{12} + 2x_{21} + 6x_{22}$$

$$S.t.: x_{11} + x_{12} \le 6$$

$$x_{21} + x_{22} \le 8$$

$$x_{11} + x_{21} \ge 9$$

$$x_{12} + x_{22} \ge 5$$

$$x_{11}, x_{12}, x_{21}, x_{22} \ge 0$$

(Hypothetical Example)

Solution

Let $x_{11} = x_1$, $x_{12} = x_2$, $x_{21} = x_3$, and $x_{22} = x_4$. Then the problem becomes

$$\min .z = -2x_1^2 - 5x_2^2 - x_3^2 - 2x_4^2 + 4x_1 + 3x_2 + 2x_3 + 6x_4$$

$$S.t.: x_1 + x_2 \le 6$$

$$x_3 + x_4 \le 8$$

$$x_1 + x_3 \ge 9$$

$$x_2 + x_4 \ge 5$$

$$x_1, x_2, x_3, x_4 \ge 0$$

To make the objective function of the problem to be linear, we employ the objective function of Equation (6) to obtain Equation (7)

$$\min .z = -2(0.25x_{11} + x_{12}) - 5(0.25_{21} + x_{22}) - (0.25x_{31} + x_{32}) - 2(0.25x_{41} + x_{42}) + 4x_1 + 3x_2 + 2x_3 + 6x_4$$

$$\min .z = -0.5x_{11} - 2x_{12} - 1.25_{21} - 5x_{22} - 0.25x_{31} - x_{32} - 0.5x_{41} - 2x_{42} + 4x_1 + 3x_2 + 2x_3 + 6x_4$$

Hence, using Equation (6), the NLPP becomes LPP as shown in Equation (7)

$$\left.\begin{aligned}
&\min . f=-0.5x_{11}-2x_{12}-1.25x_{21}-5x_{22}-0.25x_{31}-x_{32}-0.5x_{41}-2x_{42}+4x_1+3x_2+2x_3+6x_4\\
&S.t.: x_1+x_2\le 6\\
&x_3+x_4\le 8\\
&x_1+x_3\ge 9\\
&x_2+x_4\ge 5\\
&x_{11}+x_{12}-x_1=0\\
&x_{21}+x_{22}-x_2=0\\
&x_{31}+x_{32}-x_3=0\\
&x_{41}+x_{42}-x_4=0\\
&0\le x_1\le 8\\
&0\le x_2\le 8\\
&0\le x_3\le 8\\
&0\le x_4\le 8
\end{aligned}\right\}\cdots(7)$$

Solving this problem in Wolfram Mathematica yields the optimal solution:
$x_{11}=0, x_{12}=1, x_{21}=0, x_{22}=5, x_{31}=0, x_{32}=8, x_{41}=0, x_{42}=0, x_1=1, x_2=5, x_3=8, x_4=0$ with the objective value of

$$f=-2(1)^2-5(5)^2-(8)^2-2(0)^2+4(1)+3(5)+2(8)+6(0)=-156$$

REAL LIFE EXAMPLE

The flour mills limited is a manufacturing company located in Port-Harcourt. They produce Golden Penny Flour (GPF), Golden Penny Semovita (GPS), Wheat Offals (WO) etc. These products are supplied to the following states (locations) Bayelsa, Onitsha, Port Harcourt, Kano, Aba, Enugu etc. For the purpose of this thesis, only four (4) of these demand points will be considered; Enugu, Onitsha, Bayelsa and Aba. The estimated supply capacities of the three products, the demand requirements at the four sites (states) and the transportation cost per bag of each product with their percentage discounts allowed on each transported product i from the source to each of the destinations are given below.
(Extracted from Ekezie and Opara, 2013):

	Aba	Enugu	Onitsha	Bayelsa	Supply
GPF	8	20	16	10	17000
WO	6	14	16	12	11000
GPS	8	10	6	12	15000
Demand	12000	13000	11000	7000	43000

(p_{11}, p_{12}, p_{13}, p_{14}, p_{21}, p_{22}, p_{23}, p_{24}, p_{31}, p_{32}, p_{33}, p_{34}) = (0.02, 0.05, 0.03, 0.025, 0.01, 0.03, 0.035, 0.02, 0.025, 0.03, 0.015, 0.02)

The problem is to determine how many bags of each product to be transported from the source to each destination on a monthly basis in order to minimize the total transportation cost.

Solution

If we suppose that discount is given on each bag transported from i to j, then the non linear transportation problem can be formulated as:

$$\min \sum_{i=1}^{3}\sum_{j=1}^{4} c_{ij}x_{ij}$$

Subject to: $x_{11}+x_{12}+x_{13}+x_{14}\le 17$

$$x_{21}+x_{22}+x_{23}+x_{24}\le 11$$

$$x_{31}+x_{32}+x_{33}+x_{34}\le 15$$

$$x_{11}+x_{21}+x_{31}\geq 12$$
$$x_{12}+x_{22}+x_{32}\geq 13$$
$$x_{13}+x_{23}+x_{33}\geq 11$$
$$x_{14}+x_{24}+x_{34}\geq 7$$

where

$$c_{11}x_{11}=8x_{11}-p_{11}x_{11}^2 \quad c_{12}x_{12}=20x_{12}-p_{12}x_{12}^2 \quad c_{13}x_{13}=16x_{13}-p_{13}x_{13}^2 \quad c_{14}x_{14}=10x_{14}-p_{14}x_{14}^2$$
$$c_{21}x_{21}=6x_{21}-p_{21}x_{21}^2\, c_{22}x_{22}=14x_{22}-p_{22}x_{22}^2\, c_{23}x_{23}=16x_{23}-p_{23}x_{23}^2 \quad c_{24}x_{24}=12x_{24}-p_{24}x_{24}^2$$
$$c_{31}x_{31}=8x_{31}-p_{31}x_{31}^2\, c_{32}x_{32}=10x_{32}-p_{32}x_{32}^2 \quad c_{33}x_{33}=6x_{33}-p_{33}x_{33}^2 \quad c_{34}x_{34}=12x_{34}-p_{34}x_{34}^2$$

Let $x_{11}=x_1$, $x_{12}=x_2$, $x_{13}=x_3$, $x_{14}=x_4$, $x_{21}=x_5$, $x_{22}=x_6$, $x_{23}=x_7$, $x_{24}=x_8$, $x_{31}=x_9$, $x_{32}=x_r$, $x_{33}=x_q$, and $x_{34}=x_w$.

Then the problem becomes

$$\min f=8x_1-0.02x_1^2+20x_2-0.05x_2^2+16x_3-0.03x_3^2+10x_4-0.025x_4^2+6x_5-0.01x_5^2+14x_6-0.03x_6^2+16x_7$$
$$-0.035x_7^2+12x_8-0.02x_8^2+8x_9-0.025x_9^2+10x_r-0.02x_r^2+6x_q-0.015x_q^2+12x_w-0.02x_w^2$$

Subject to: $x_1+x_2+x_3+x_4\leq 17$

$$x_5+x_6+x_7+x_8\leq 11$$
$$x_9+x_r+x_q+x_w\leq 15$$
$$x_1+x_5+x_9\geq 12$$
$$x_2+x_6+x_p\geq 13$$
$$x_3+x_7+x_q\geq 11$$
$$x_4+x_8+x_w\geq 7$$
$$x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_r,x_q,x_w\geq 0$$

To make the objective function of the problem linear, we employ the objective function of Equation (6) to obtain Equation (8)

$$\left.\begin{array}{l}
\min. f = 8x_1 - 0.005x_{11} - 0.02x_{12} + 20x_2 - 0.0125x_{21} - 0.05x_{22} + 16x_3 - 0.0075x_{31} - 0.03x_{32} + 10x_4 - 0.00625x_{41} \\
- 0.025x_{42} + 6x_5 - 0.0025x_{51} - 0.01x_{52} + 14x_6 - 0.0075x_{61} - 0.03x_{62} + 16x_7 - 0.00875x_{71} - 0.035x_{72} + 12x_8 \\
- 0.005x_{81} - 0.02x_{82} + 8x_9 - 0.00625x_{91} - 0.0025x_{92} + 10x_r - 0.005x_{r1} - 0.02x_{r2} + 6x_q - 0.00375x_{q1} - 0.015x_{q2} + 12x_w \\
- 0.005x_{w1} - 0.02x_{w2} \\
Sub.to: x_1 + x_2 + x_3 + x_4 \le 17 \\
x_5 + x_6 + x_7 + x_8 \le 11 \\
x_9 + x_r + x_q + x_w \le 15 \\
x_1 + x_5 + x_9 \ge 12 \\
x_2 + x_6 + x_r \ge 13 \\
x_3 + x_7 + x_q \ge 11 \\
x_4 + x_8 + x_w \ge 7 \\
x_{11} + x_{12} - x_1 = 0 \\
x_{21} + x_{22} - x_2 = 0 \\
x_{31} + x_{32} - x_3 = 0 \\
x_{41} + x_{42} - x_4 = 0 \\
x_{51} + x_{52} - x_5 = 0 \\
x_{61} + x_{62} - x_6 = 0 \\
x_{71} + x_{72} - x_7 = 0 \\
x_{81} + x_{82} - x_8 = 0 \\
x_{91} + x_{92} - x_9 = 0 \\
x_{r1} + x_{r2} - x_r = 0 \\
x_{q1} + x_{q2} - x_q = 0 \\
x_{w1} + x_{w2} - x_w = 0 \\
0 \le x_i \le 17, i = 1,2,3,4,5,6,7,8,9,r,q,w
\end{array}\right\} \ldots(8)$$

Solving this problem in Wolfram Mathematica yields the optimal solution:

Hence, the optimal solution becomes $x_1 = 10$, $x_4 = 7$, $x_5 = 2$, $x_6 = 9$, $x_r = 4$, and $x_q = 11$ with the objective value as:

$$f = 8(10) - 0.02(10)^2 + 10(7) - 0.025(7)^2 + 6(2) - 0.01(2)^2 + 14(9) - 0.03(9)^2 + 10(4) - 0.02(4)^2 + 6(11) - 0.015(11)^2 = 386.17$$

Hence the following allocation should be made: 10000 bags of GPF should be supplied to Aba, 7000 of the same product should be supplied to Bayelsa. 2000 bags of WO should be allocated to Aba, and 9000 bags of the same product to Enugu. Again, allocate 4000 bags of GPS to Enugu and 11000 bags of the same product to Onitsha.

Conclusion

The study has developed an algorithm that can be employed to tackle optimization problems with nonlinear transportation problem. The algorithms used two breakpoints. After transforming the nonlinear objective function to linear function, Wolfram Mathematica was employed to get the optimal solution. The two problems solved in this study showed that the proposed algorithm converged faster than Wolfram Mathematica, though with the same objective values

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