Alpha Level Subgroups of Alpha-Fuzzy Subgroup

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Abstract
The concept of α-level subgroup of an α-fuzzy subgroup is defined and its characterizations are obtained. It is also proved a necessary and sufficient condition for two α-level subgroups to be equal.

Keywords: fuzzy subgroup,α-fuzzy set, α-fuzzy subgroup, level subgroups.

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I. Introduction and Preliminaries

In 1965, Zadeh introduced the notion of fuzzy set[1]. In 1971, A. Rosenfeld defined fuzzy groups[2] and many group theory results have been extended to fuzzy groups. P.Sivaramakrishna Das introduced the concept of level subgroups of a fuzzy group[3]. P.K.Sharma introduced the notion of α-fuzzy set, α-fuzzy subgroup, α-fuzzy coset and analysed their characterizations in[5]. In this paper, α- level subgroups of an α-fuzzy

subgroup is defined and its characterizations are obtained. It is also discussed about α-fuzzy subgroups relative to a finite cyclic group.

Throughout this paper, α- denotes a member of [0,1].

Definition 1.1 Let X be a non empty set. A **fuzzy subset** A of X is a function

$A : X \to [0,1]$.

Definition 1.2 A fuzzy subset A of a group G is called a **fuzzy subgroup** of G if

(i) $A(xy) \geq \min\{A(x), A(y)\}$

(ii) $A(x^{-1}) = A(x)$, for all $x, y \in G$.

Definition 1.3 Let A be a fuzzy subset of a set S. For $t \in [0,1]$, the set $A_t = \{x \in S / A(x) \geq t\}$ is called a **level subset** of A.

Definition 1.4 Let G be a group and A be a fuzzy subgroup of G. The subgroups A_t, $t \in [0,1]$ and $t \leq A(e)$, are called **level subgroups** of A.

Definition 1.6 Let A be a fuzzy subset of a group G. Let $\alpha \in [0,1]$. Then an **α-fuzzy subset** of G(with respect to a fuzzy set A), denoted by A^α, is defined as $A^\alpha(x) = min\{A(x), \alpha\}$, for all $x \in G$.

Definition 1.7 A fuzzy subset A of a group G is called a **α-fuzzy subgroup** of G if

(i) $A^\alpha(xy) = min\{A^\alpha(x), A^\alpha(y)\}$

(ii) $A^\alpha(x^{-1}) = A^\alpha(x)$, for all $x, y \in G$.

Theorem 1.8 If $A: G \to [0,1]$ is a α-fuzzy group of a group G, then

(i) $A^\alpha(x) \le A^\alpha(e)$, where e is the identity element of G.

(ii) $A^\alpha(xy^{-1}) = A^\alpha(e) \Rightarrow A^\alpha(x) = A^\alpha(y)$, for all $x, y \in G$.

Theorem 1.9 $A: G \to [0,1]$ is a fuzzy group of a group G iff $A^\alpha(xy^{-1}) \ge \min\{A^\alpha(x), A^\alpha(y)\}$, for all $x, y \in G$.

II. α- LEVEL SUBGROUPS OF α-FUZZY SUBGROUP

In this section, the concept of α- level subgroup of an α-fuzzy subgroup is defined and its properties are determined. The relation between α-fuzzy subgroups and their α-level subgroups is obtained.

Definition 2.1. Let G be a group. Let $A : G \to [0,1]$ be a fuzzy subset of G. For $t \in [0,1]$, an α-level subset of the fuzzy subset A, denoted by A_t^α, is defined as $A_t^\alpha = \{x \in G | A^\alpha(x) \ge t\}$.

Example 2.2. Let $G = \{e, a, b, ab\}$ be a group. Let $A : G \to [0,1]$ be defined as

$$A(x) = \begin{cases} 0.6, \text{ if } x = e \\ 0.8, \text{if } x = a, b \\ 0.7, \text{if } x = ab \end{cases}$$

Let α = 0.7 and t = 0.65. Then $A_t^\alpha = \{a, b, ab\}$ is an α-level subset of A.

Theorem 2.3 Let G be a group. Let $t \in [0,1]$ and let $t \le A_t^\alpha(e)$. Let $A:G \to [0,1]$ be an α-fuzzy subgroup of a group G. Then the α-level subset A_t^α is a subgroup of G, where e is the identity of G.

Proof:

Let G be a group. Let $A : G \to [0,1]$ be an α-fuzzy subgroup. Let $t \in [0,1]$ and

$t \le A_t^\alpha(e)$. $A_t^\alpha = \{x \in G | A^\alpha(x) \ge t\}$ is non empty subset of G. If $x, y \in A_t^\alpha$, then $\min\{A^\alpha(x), A^\alpha(y)\} \ge t$. Since A^α is a fuzzy subgroup, $A^\alpha(xy) \ge \min\{A^\alpha(x), A^\alpha(y)\} \ge t$ which implies $xy \in A_t^\alpha$. Also $A^\alpha(x^{-1}) = A^\alpha(x)$. Since $x \in A_t^\alpha$, $A^\alpha(x) \ge$ t which implies that $A^\alpha(x^{-1}) \ge$ t. Therefore $x^{-1} \in A_t^\alpha$.Thus A_t^α is a subgroup of G.

Theorem 2.4: Let G be a group. Let A: $G \to [0,1]$ be a fuzzy subset of G such that A_t^α is a subgroup of G, for all $t \in [0,1]$, $t \le A^\alpha$(e). Then A is an α- fuzzy subgroup of G.

Proof:

Let $x,y \in G$ and let $A^{\alpha}(x) = t_1$ and $A^{\alpha}(y) = t_2$, where $t_1, t_2 \in [0,1]$. Therefore $x \in A^{\alpha}_{t_1}$ and $y \in A^{\alpha}_{t_2}$. Suppose that $t_1 < t$. Then $A^{\alpha}_{t_2} \subset A^{\alpha}_{t_1}$ which implies that $y \in A^{\alpha}_{t_1}$. Since $x, y \in A^{\alpha}_{t_1}$ and $A^{\alpha}_{t_1}$ is a subgroup of G, $xy \in A^{\alpha}_{t_1}$.Therefore $A^{\alpha}(xy) \geq t = \min\{A^{\alpha}(x), A^{\alpha}(y)\}$. Now let $x \in G$ and $A^{\alpha}(x)=t$. Therefore $a \in A^{\alpha}_t$ which implies that $a^{-1} \in A^{\alpha}_t$, since A^{α}_t is a subgroup of G. Then $A^{\alpha}(x^{-1}) \geq t = A^{\alpha}(x)$ Thus A^{α} is a fuzzy group and hence A is an α-fuzzy subgroup of G.

Definition 2.5 Let G be a group. Let A be an α-fuzzy subgroup of G. If $t \in [0,1]$ and $t \leq A^{\alpha}(e)$, then the subgroups A^{α}_t are said to be α-level subgroups of A .

Example 2.6

Let $G = \{e, a, b, c\}$ be the klein four group. Let a fuzzy subset A of G be defined as

$$A(x) = \begin{cases} 1, & \text{if } x = e, a \\ 3/4, & \text{if } x = b, c \\ 0, & \text{otherwise} \end{cases}$$

If $\alpha = 0.85$, then A is an α-fuzzy subgroup of G. If $t = 0.8$, then $A^{\alpha}_t = \{e, a\}$ which is a subgroup of G and is an α-level subgroup of A.

Remark 2.7 If G is a group which is also finite, then the number of subgroups of G is finite whereas the number of α-level subgroups of an α-fuzzy subgroup A seems to be infinite. But since every α-level subgroup is a subgroup of G, not all these α- level subgroups will be distinct. This is characterized in the following theorem.

Theorem 2.8 Let G be a group. Let A be an α-fuzzy subgroup of G. Let $t_1, t_2 \in [0,1]$ and $t_1 < t_2$. Then two α-level subgroups $A^{\alpha}_{t_1}, A^{\alpha}_{t_2}$ of A are equal iff there is no $x \in G$ such that $t_1 < A^{\alpha}(x) < t_2$.

Proof:

Assume that $A^{\alpha}_{t_1}, = A^{\alpha}_{t_2}$. Suppose that there exists $x \in G$ such that $t_1 < A^{\alpha}(x) < t_2$. Then $x \in A^{\alpha}_{t_1}$ But $x \notin A^{\alpha}_{t_2}$. This implies that $A^{\alpha}_{t_1}, \neq A^{\alpha}_{t_2}$ which is a contradiction. Conversely, assume that there is no $x \in G$ such that $t_1 < A^{\alpha}(x) < t_2$. Since $t_1 < t_2$, $A^{\alpha}_{t_2} \subset A^{\alpha}_{t_1}$. If $x \in A^{\alpha}_{t_1}$, then $t_1 \leq A^{\alpha}(x)$. Also $t \leq A^{\alpha}(x)$ does not lie between t_1 , t_2. Therefore $t_2 \leq A^{\alpha}(x)$ which implies that $A^{\alpha}_{t_1}, = A^{\alpha}_{t_2}$.

Corollary 2.9 Let G be a finite group of order n. Let A be an α-fuzzy subgroup of G. Let t_i 's be the distinct images of elements of G . Then $A^{\alpha}_{t_i}$'s are the only α-level subgroup of A [$im(A^{\alpha}) = \{t_i / A^{\alpha}(x) = t_i$ for some $x \in G$]

Proof:

Since $t_i \leq A^{\alpha}(e)$, for all t_i where $t_i \in [0,1]$, e -identity of G By theorem 2.3, $A^{\alpha}_{t_i}$ is a subgroup of G for each i. Let $t \in [0,1]$. Suppose that $t = A^{\alpha}(e)$, then $A^{\alpha}_{t} = \{e\}$ which is an $\alpha -$level subgroup of A. Let $t \neq Im(A^{\alpha})$. Suppose that $t_i < t < t_j$,where $t_i, t_j \in \mathrm{Im}(A^{\alpha})$. Since $A^{\alpha}(x) \neq t$, for all $x \in G$ by theorem 2.8, $A^{\alpha}_{t_i} = A^{\alpha}_{t_j} = A^{\alpha}_{t}$ which is an $\alpha -$level subgroup of A. Suppose that $t < t_r$, where t_r is the least element in $\mathrm{Im}(A^{\alpha})$. Then $A^{\alpha}(x) > t$ for all $x \in G$. Therefore $A^{\alpha}_{t_r} = A^{\alpha}_{t} = G$. Thus for any $t \in [0,1]$, $A^{\alpha}_{t_i}$'S are the only α-level subgroup of A.

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