Some Eulerian Polynomials of Higher Order

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Abstract - *In the current paper we will derive identities of Eulerian polynomials of higher order from non linear ordinary differential equations. We will show that the generating functions of Eulerian polynomials are the solutions of our non linear ordinary differential equations.*

Keywords - *Eulerian numbers, polynomials.*

Introduction

We know that the generating function $z(x,y)$ of Euler polynomials $E_m(y)$ is given by

$$Z(x,y)=\frac{2}{e^x+1}e^{yx}=\sum_{m=0}^{\infty}E_m(y)\frac{x^m}{m!} \tag{1}$$

For the special case $y=0, E^m\ (0)=E_m$ is the mth Euler number. From (1), we note that

$$E_0=1,\ \ (E+1)^m+E_m=0, \quad if \quad n>0, \tag{2}$$

With the usual convention of replacing E^n by E_n. The generating function $Z_v(x,y)$ of Eulerian polynomials $J_m(x/v)$ are defined by

$$Z_v(x,y)=\frac{1-v}{e^x-v}e^{yx}=\sum_{m=0}^{\infty}J_m(x/v)\frac{x^m}{m!} \tag{3}$$

Where $v\in C$ with $v\neq 1$

For the special case $y=0,\ \ J_m(0/v)=J_m(v)$.

$J_m(v)$ is called the mth Eulerian number. Sometimes it is called the mth Frobenius Euler number.

From equation (1) and equation (3). We can note that $J_m(y/-1)=E_m(y)$

From equation (3), we get

$$\frac{1-v}{e^x-v}e^{yx}=\left[\sum_{i=0}^{\infty}J_i(v)\frac{x^v}{i!}\right]\left[\sum_{j=0}^{\infty}\frac{y^jx^j}{j!}\right] \tag{4}$$

From equation (3) and equation (4), we get

$$J_m(x/v)=\sum_{i=0}^{m}\binom{m}{i}x^{m-i}J_i(v)=(J(v)+y)^m \tag{5}$$

Now replacing $J^m(v)\ \ by\ \ J_m(v)$ in equation (5), we get

$$1-v=\frac{1-v}{e^x-v}e^x-\frac{1-v}{e^x-v}v=\sum_{m=0}^{\infty}(J(v)+1)^m\frac{x^m}{m!}-\sum_{n=0}^{\infty}vJ_m(v)\frac{x^m}{m!} \tag{6}$$

Hence, we get the recurrence relation for $J_m(v)$.

$$J_0(v)=1,\quad J_m(1/v)-vJ_m(v)=(1-v)\delta_0,m \tag{7}$$

Where $\delta_{m,k}$ is kronecker symbol

For $M \in M$, the m^th^ Eulerain polynomials $J_m^{(M)}(x/v)$ of order M are defined be generating function as follows

$$Z_v^M(x,y)=\left(\frac{1-v}{e^x-v}\right)x\left(\frac{1-v}{e^x-v}\right)x...x\left(\frac{1-v}{e^x-v}\right)e^{yx}=\sum_{m=00}^{\infty}J_m^{(M)}(x/v)\frac{x^m}{m!} \tag{8}$$

For the special case $y=0,\ J_m^{(M)}(0/v)=J_m^{(M)}(v)$ are called the m^th^ Eulerian number of order M.

NON –LINER DIFFERENTIAL EQUATIONS

We define that $Z=Z(v)=\dfrac{1-v}{e^x-v}$

$$Z^M(x,y)=Z\quad x...x\quad Ze^{yx}\text{ for } M\in M \tag{9}$$

We have $Z(x,y)=Z_v(x,y)=Ze^{yx}$ from equation (9), we get

$$Z^{(1)}=\frac{dZ}{dv}=-\frac{1}{1-v}\frac{1-v}{e^x-v}+\frac{1}{1-v}\left(\frac{1-v}{e^x-v}\right)^2 \tag{10}$$

From equation (10), we get

$$Z^{(1)}(x,y)=Z^{(1)}e^{xy}=-\frac{1}{1-v}(Z(x,y)-Z^2(x,y)), \tag{11}$$

$$(1-v)Z^{(1)}+Z=Z^2$$

Theorem 1. For $v\in C$ with $v\neq 1,\ M\in M$

$Z(v)=\dfrac{1-v}{e^x-v}$ is a solution of

$$Z^M(v)=\sum_{k=0}^{M-1}\frac{1}{k!}(1-v)^kZ^{(k)}(v) \tag{12}$$

Where $Z^{(k)}(v)=\dfrac{d^kZ(v)}{dv^k}$ *and* $Z^M(v)=Z(v)x...xZ(v)$

Proof:-

We will prove by induction

(i) If M=1, the it is obvious

(ii) Assume that equation (12) is true for some M>1.

Let us suppose

$$MZ^{M-1}Z^{(1)} = \sum_{k=0}^{M-1} \frac{1}{k!}\left(\left(-k(1-v)^{k-1}Z^{(k)} + (1-v)^k Z^{(k+1)}\right)\right.$$

$$= \frac{1}{(M-1)!}(1-v)^{M-1}Z^{(M)} \tag{13}$$

From equation (11) and equation(13), we get

$$\frac{1}{M!}(1-v)^M Z^{(M)} = Z^{M-1}(1-v)Z^{(1)} = Z^{N-1}(-Z+Z^2) \tag{14}$$

From equation (13) and equation (14), we have

$$Z^{M+1} = Z^M + \frac{1}{M!}(1-v)^M Z^{(M)} = \sum_{k=0}^{M} \frac{1}{k!}(1-v)^k Z^{(k)}$$

Corollary1. For $v \in C$ with $v \neq 1, M \in M$,

$Z(x,y) = \dfrac{1-v}{e^x - v} e^{yx}$ is solution of the

$$Z^N(x,y) = \sum_{k=0}^{N-1} \frac{1}{k!}(1-v)^k Z^{(k)}(x,y) \tag{15}$$

It is obvious proved from the fact that

$$Z^M(x,y) = Z^M(v)e^{yx} \quad and \quad Z^{(k)}(x,y) = \frac{d^k Z(v)}{dv^k} e^{yx}$$

IDENTITIES OF EULERIAN NUMBERS AND POLYNOMIALS OF HIGHER ORDER

Therom2. For $M \in M, m \in Z = M \cup \{0\}, we \quad have$

$$J_m^{(M)}(v) = \sum_{k=0}^{M-1} \frac{1}{k!}(1-v)k \frac{d^k J_m(v)}{dv^k}$$

Proof :-

From equation (8) and equation (9), we get

$$Z^M = \frac{1-v}{e^x - v} x \ldots x \frac{1-v}{e^x - v} = \sum_{m=0}^{\infty} J_m^{(M)}(v) \frac{x^m}{m!} \tag{16}$$

From equation (3) and equation (9), we get

$$Z^{(k)} = \frac{d^k Z(v)}{dv^k} = \sum_{m=0}^{\infty} \frac{d^k J_m(v)}{du^k} \frac{x^m}{m!} \tag{17}$$

From equation (15) and comparing with coefficients of equation (16) and (17), we obtain the required result of this theorem.

Corrolary2. For $M \in M, m \in Z_+$ we have

$$\sum_{i_1+\ldots+i_M=m}\binom{m}{i_1,\ldots,i_M} Ji_1(v)Ji_2(v)\ldots Ji_N(v)=\sum_{k=0}^{M-1}\frac{1}{k!}(1-v)^k\frac{d^kJ_m(v)}{dv^k}$$

Proof. We have

$$\sum_{m=0}^{\infty}J_m^{(M)}(v)\frac{x^m}{m!}=\frac{1-v}{e^x-v}x\ldots x\frac{1-v}{e^x-v}=\left[\sum_{i_1}^{\infty}J_{i_1}(v)\frac{x^{i_1}}{i!}\right]x\ldots x\left[\sum_{i_M=0}^{\infty}J_{i_1}(v)\frac{x^{i_M}}{i_M!}\right] \tag{18}$$

$$=\sum_{m=0}^{\infty}\left[\sum_{i_1+\ldots+i_M=m}\binom{m}{i_1,\ldots,i_M}J_{i_1}(v)J_{i_2}(v)\ldots J_{i_M}(v)\right]\frac{x^m}{m!}$$

Hence proved.

Corollary 3. For $M\in M, m\in Z_+$, we have

$$J_m^{(M)}(x/v)=\sum_{k=0}^{M-1}\frac{1}{k!}(1-v)^k\sum_{i=0}^{m}\binom{m}{i}x^{m-i}\frac{d^kJ_m(v)}{dv^k}$$

Proof. From equation (4) and (17), we have

$$Z^{(k)}(x,y)=Z^{(k)}e^{yx}$$

$$=\left[\sum_{m=0}^{\infty}\frac{d^kJ_m(v)}{dv^k}\frac{x^m}{m!}\right]\left[\sum_{m=0}^{\infty}\frac{y^mx^m}{m!}\right]$$

$$=\sum_{m=0}^{\infty}\left[\sum_{i=o}^{m}\binom{m}{i}y^{m-i}\frac{d^kJ_i(v)}{dv^k}\right]\frac{x^m}{m!} \tag{19}$$

From equation (8),(15) and (19).

Hence proved.

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