A Note On Ideal Signed Graph

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Abstract: *In this paper we define Ideal signed graph of a signed graph and given some characterization.*

Keywords: *signed graphs, Marked graphs, balance, Switching, Antipodal graph.*

Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [2]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self-loops and isolates.

In 1953, Harary published ``on the notion of balance of a signed graph''[3], the first paper to introduce signed graphs. In this paper, Harary defined a signed graph as a graph whose edge set has been partitioned into positive and negative edges. He called a cycle positive if it had an even number of negative edges, and he called a signed graph balanced if every cycle is positive. Then he gave both necessary and sufficient conditions for balance.

Since then, mathematicians have written numerous papers on the topic of signed graphs. Many of these papers demonstrate the connection between signed graphs and different subjects: circuit design, coding theory, physics and social psychology (Abelson and Rosenberg [5]). While these subjects seem unrelated, balance plays an important role in each of these fields.

Four years after Harary's paper, Abelson and Rosenberg, wrote a paper in which they discuss algebraic methods to detect balance in a signed graphs. It was one of the first papers to propose a measure of imbalance, the ``complexity'' (which Harary called the ``line index of balance''). Abelson and Rosenberg introduced an operation that changes a signed graph while preserving balance and they proved that this does not change the line index of imbalance. For more new notions on signed graphs refer the papers [7,6]

A signed graph is an ordered pair $\Gamma = (G, \sigma)$, where $G = (V, E)$ is a graph called *underlying graph* of Γ and $\sigma : E \rightarrow \{+,-\}$is a function. A *marking* of Γ is a function $\mu : V(G) \rightarrow \{+,-\}$, A signed graph Γ together with a *marking* μ is denoted by Γ_μ. Given a signed graph Γ one can easily define a marking μ of Γ as follows:
For any vertex $v \in V(\Gamma)$,

$$\mu(v) = \prod_{e \in E(\Gamma)} \sigma(uv)$$

The marking μ of Γ is called *canonical marking* of Γ. In a signed graph $\Gamma = (G, \sigma)$, for any $D \subseteq E(G)$ the sign $\sigma(D)$ is the product of the signs on the edges of D.

The following characterization of balanced signed graphs is well known.

Proposition1:(E. Sampathkumar [10]) A signed graph $\Gamma = (G, \sigma)$ is balanced if, and only if, there exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \mu(u)\mu(v)$.

Let $\Gamma = (G, \sigma)$ be a signed graph. Consider the marking μ on vertices of Γ defined as follows: each vertex $v \in V$, $\mu(v)$ is the product of the signs on the edges incident at v.
Complement of Γ is a signed graph $\underline{\Gamma} = (\underline{G}, \sigma')$, where for any edge $e = uv \in \underline{G}$, $\sigma'(uv) = \mu(u)\mu(v)$. Clearly, $\underline{\Gamma}$ as defined here is a balanced signed graph due to Proposition 1.

The idea of switching a signed graph was introduced in [5] in connection with structural analysis of social behavior and also its deeper mathematical aspects, significance and connections maybe found in [8].

Switching Γ with respect to a marking μ is the operation of changing the sign of every edge of Γ to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by $(\Gamma)_\mu(\Gamma)$ and is called μ-switched signed graph or just switched signed graph. Two signed graphs $\Gamma_1 = (G, \sigma)$ and $\Gamma_2 = (G', \sigma')$ are said to be isomorphic, written as $\Gamma_1 \cong \Gamma_2$ if there exists a graph isomorphism f: $G \to G'$ (that is a bijection f:$V(G) \to V(G')$to such that if uv is an edge in G then f(u)f(v) is an edge in G' such that for any edge $e \in G$, $\sigma(e) = \sigma'(f(e))$. Further a signed graph $\Gamma_1=(G, \sigma)$ switches to a signed graph $\Gamma_2=(G', \sigma')$ (or that Γ_1 and Γ_2 are switching equivalent) written $\Gamma_1 \sim \Gamma_2$, whenever there exists a marking μ of Γ_1 such that $\Gamma_\mu(\Gamma_1) \cong \Gamma_2$. Note that $\Gamma_1 \sim \Gamma_2$ implies that $G \cong G'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $\Gamma_1 = (G, \sigma)$ and$\Gamma_2 = (G', \sigma')$ are said to be weakly isomorphic or cycle isomorphic [1] if there exists an isomorphismϕ $G \to G'$ such that the sign of every cycle Z in Γ_1 equals to the sign of ϕ(Z) in Γ_2.

The following result is well known [1]:

Proposition 2(T. Zaslavsky) [1]:
Two signed graphs Γ_1 and Γ_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.

Antipodal Graph
Singleton (1968) introduced the concept of the Antipodal graph of a graph G, denoted by A(G), is the graph on the same vertices as of G, two vertices being adjacent if the distance between them is equal to the diameter of G.

Antipodal Signed Graph

Antipodal signed graph of a signed graph is a signed graph whose underlying graph is A(G) and sign of any edge uv in A(Γ)is μ(u)μ(v),where $\mu$is the canonical marking of Γ. Further, a signed graph is called Antipodal signed graph, if for some signed graph Γ is called antipodal signed graph, if $\Gamma \cong A(\Gamma')$ for some signed graph Γ'. The following result indicates the limitations of the notion A (Γ) as introduced above, since the entire class of unbalanced signed graphs is forbidden to be antipodal signed graphs.

The following result indicates the limitations of the notation as introduced above.

Proposition 3: For any signed graph $\Gamma = (G, \sigma)$, its Antipodal Signed graph A (Γ) is balanced.

Ideal Graph of a graph [9]
For a graph G with sets C of cycles, l of longest paths with all the internal vertices of degree 2, and $U \subset C$, $V \subset l$, it's ideal graph $I_d^{U,V}(G)$ of the graph G is formed as follows

(i) These cycles and the edges lying on a cycle in U or $C \setminus U$ will remain or not same in ideal graph $I_d^{U,V}(G)$ of G.

(ii) Every longest u-v path in V of $l \setminus V$ is considered as an edge uv or not ideal graph $I_d^{U,V}(G)$ of G.

Proposition 4: **[9]** A graph G is isomorphic to the graph G' if and only if I_d (G) is isomorphic to I_d (G') and isomorphic edges having same vanishing number.

From above definition we found a new notation

Ideal Signed Graph

The Ideal signed graph I_d (Γ) of a signed graph $\Gamma = (G, \sigma)$ is a signed graph whose underlying graph is I_d (G) and sign of every edge uv in I_d (Γ) is μ(u)μ(v).

Proposition 5: For any balanced signed graph Γ, $I_d(\Gamma)$ is balanced.

Proof: Sign of any edge in uv in $I_d(\Gamma)$ is μ(u)μ(v) where μ is canonical marking of Γ. By the proposition $I_d(\Gamma)$ is balanced.

Corollary 6: For any consistent signed graph Γ, $I_d(\Gamma)$ is consistent.

Proposition 7:

1. $I_d(P_n) = p_2$
2. $I_d(c_n) = c_2$, $I_d(w_n) = w_n$, $I_d(K_n) = K_n$
3. $I_d(G) = p_2$
4. $I_d(K_{m,n}) = K_{m,n}$ except $K_{1,2}$
5. $I_d(G) = G$ if $\delta \geq 3$
6. $I_d(G) = G$ if G is Eulerian.
7. $I_d(I_d(G)) = I_d(G)$

From the above proposition we have the following.

Proposition 8:

1. $I_d(\Gamma) \sim \Gamma$, if and only if, there underlying graph $G = K_n$.
2. $I_d(\Gamma) \sim \Gamma$, if and only if there underline graph G is Eularian graph.

Proof:
From the above definitions we can prove.

Corollary 9: $I_d(I_d(\Gamma)) = I_d$

Proposition 10:
For any two balanced signed graphs Γ_1 and Γ_2 with the same underlying graphs, there Ideal signed graphs are switching equivalent.
Proof:
$\Gamma_1 = (G, \sigma)$ and $\Gamma_2 = (G', \sigma)$ are the two balanced signed graphs with $G \cong G'$ and hence result follows from proposition 1 and proposition 2.

Proposition 11:
For any balanced signed graph Γ, $I_d[\eta(\Gamma)] \sim I_d(\Gamma)$ if and only if $\eta(\Gamma) \sim I_d(\Gamma)$.

Proof:
Result follows from proposition 10.

Proposition 12:
For any signed graph $\Gamma = (G, \sigma)$, $I_d(\Gamma)$ is balanced if and only if $\Gamma = (G, \sigma)$, $I_d(\Gamma)$ is balanced if and only if Γ is an antipodal signed graph.

Proof:
If Γ is an antipodal graph, from proposition 3, Γ is balanced so from proposition 5, $I_d(\Gamma)$ is balanced.
If $I_d(\Gamma)$ is balanced and there underlining graph is antipodal then there exists a graph H' such that we can consider a signed graph $\Gamma' = (H', \sigma)$ from the definition of antipodal signed graph, and by proposition 3,
$I_d[A(\Gamma')] \cong I_d(\Gamma)$ therefore $I_d(\Gamma)$ is antipodal.

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