ClosedLieIdeals of Prime Rings with Generalizedα−derivations

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Abstract — *Inthispaper, westudy on closedLieideals* $\boldsymbol{L}$ *of prime ring* $\boldsymbol{R}$ *with generalized* $\boldsymbol{\alpha}-$ *derivation and show* $\boldsymbol{L} \subseteq \boldsymbol{Z}(\boldsymbol{R})$ *under several conditions. Also, we investigate commutativity of Lie ideal* $\boldsymbol{L}$ *for two generalized* $\boldsymbol{\alpha}-$ *derivations* $\boldsymbol{F}$ *and* $\boldsymbol{G}$*.*

Keywords — *Prime ring, Lie ideal, Generalized*$\boldsymbol{\alpha}-$ *derivations.*

I. INTRODUCTION

Let$Z(R)$ be center of ring R. A ring R is prime ring if $rRy = (0)$ then $r = 0$ or $y = 0$ for any $r, y \in R$. If L is an additive subgroup of R such that $[L, R] \subseteq L$, then L is a Lie ideal of R. Also, if $r^2 \in L$ for all $r \in L$, then L is square closed.

Thenotation$[r, y]$ is used for commutator $ry - yr$ and $r \circ y$ is used for anti-commutator $ry + yr$ for any $r, y \in R$. Also, following equalities hold for commutator and anti-commutator.

- $[rp, t] = r[p, t] + [r, t]p$
- $[r, pt] = [r, p]t + p[r, t]$
- $(rp) \circ t = r(pot) - [r, t]p = (r \circ t)p + r[p, t]$
- $r \circ (pt) = (r \circ p)t - p[r, t] = p(r \circ t) + [r, p]t$

From [5], an additivemapping$d$ from R into R is called derivation if $d(rp) = d(r)p + rd(p)$ for all $r, p \in R$. In [3], Bresar introduced definition of generalized derivation. An additive mapping F from R into R is called generalized derivation associated with derivation d if $F(rp) = F(r)p + rd(p)$ for all $r, p \in R$. From [1] and [4], definition of α −derivation and generalized α −derivation is given as follows: Let d from R into R be an additive mapping and α be an automorphism of R. If $d(rp) = d(r)p + \alpha(r)d(p)$ holds for all $r, p \in R$, then d is called α −derivation. Let F from R into R be an additive mapping. If $F(rp) = F(r)p + \alpha(r)d(p)$ holds for all $r, p \in R$, then F is called generalized α −derivation associated with α −derivation d. In many years, authors have proved commutative theorems for prime rings with $\alpha -$ derivation and generalized $\alpha -$ derivation. Also many researchershavegeneralizedresultstoidealsandLieideals of ring.

Inthispaper, westudy on prime ringswithgeneralized$\alpha -$ derivation. Let R be a prime ring and $charR \neq 2, L$ be a square closed Lie ideal of R and $0 \neq F: R \to R$ be a generalized $\alpha -$ derivation associated with $\alpha -$ derivation d such that $d\big(Z(L)\big) \cap Z(L) \neq (0)$ and $\alpha(L) \subset L$. We study following conditions and prove $L \subseteq Z(R)$. $(i)[F(r), r] \in Z(R)$ for all $r \in L$. $(ii)F(r) \circ r \in Z(R)$ for all $r \in L$. (iii) $[F(r), F(p)] - [r, p]$ for all $r, p \in L$. (iv) $F(r) \circ F(p) - r \circ p \in Z(R)$ for all $r, p \in L$. (v) $[F(r), F(p)] - r \circ p$ for all $r, p \in L$. (vi) $F(r) \circ F(p) - [r, p] \in Z(R)$ for all $r, p \in L$. (vii) $F[r, p] - r \circ p \in Z(R)$ for all $r, p \in L$. $(viii)$ $F(r \circ p) - [r, p] \in Z(R)$ for all $r, p \in L$. (ix) $[F(r), \alpha(p)] - [r, p] \in Z(R)$ for all $r, p \in L$. (x) $F(r) \circ \alpha(y) - r \circ p \in Z(R)$ for all $r, p \in L$.

Inaddition, weinvestigatecommutativeproperty on squareclosedLie ideal of prime ring R for two generalized $\alpha -$ derivations $0 \neq F, G: R \to R$ associated with $\alpha -$ derivations $d, g: R \to R$ respectively. We study following conditions and prove $L \subseteq Z(R)$. (i) $[F(r), G(p)] - [r, p] \in Z(R)$ for all $r, p \in L$. (ii) $F[r, p] - [p, G(r)] \in Z(R)$ for all $r, p \in L$. (iii) $F(r \circ p) - p \circ G(r) \in Z(R)$ for all $r, p \in L$. (iv) $[F(r), r] - [r, G(r)] \in Z(R)$ for all $r \in L$. (v) $F(r) \circ r - r \circ G(r) \in Z(R)$ for all $r \in L$.

II. PRELIMINARIES

Remark 1 *Let R be ring and*α *be an automorphism of ring* R*. If* $\alpha(r) \in Z(R)$*, then* $r \in Z(R)$*.*

Remark 2 *Let*R *be a prime ring. For an elements* $r \in Z(R)$ *and* $s \in R$*, if* $rs \in Z(R)$*, then* $s \in Z(R)$ *or* $r = 0$*.*

Lemma 2.1 *Let$R$ be a prime ring with char$R \neq 2$ and L be a square closed Lie ideal of R. Then $2rs \in L$ for all $r, s \in L$.*

Lemma 2.2 *[6]Lemma 2.5 Let$R$ be a $2-$torsion free semiprime ring and L be a nonzero Lie ideal of R. Then $Z(L) \subseteq Z(R)$.*

Lemma 2.3 *[2]Lemma 3.1 Let$R$ be a prime ring with $char(R) \neq 2$ and L be a nonzero square closed Lie ideal of R. If $[x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Lemma 2.4 *[2] Lemma 3.2 Let$R$ be a prime ring with $char(R) \neq 2$ and L be a nonzero square closed Lie ideal of R. If $x \circ y \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

III. RESULTS

Theorem 3.1 *Let ring R be prime with$char(R) \neq 2$, $(0) \neq L$ be a square closed Lie ideal of R and $0 \neq F: R \to R$ be a generalized $\alpha-$derivation associatedwith$\alpha-$derivation d such that $Z(L) \cap d\big(Z(L)\big) \neq (0)$ and $\alpha(L) \subset L$. If $[F(r), r] \in Z(R)$ for all $r \in L$, then R is commutative.*

*Proof.*Let$[F(r), r] \in Z(R)$ for all $r \in L$. Replacing r by $r + p$, we get

$$[F(r), p] + [F(y), p] \in Z(R) \text{ for all } r, p \in L. \tag{3.1}$$

From$Z(L) \cap d\big(Z(L)\big) \neq (0)$, we take fixed element $0 \neq s \in Z(L)$ which $0 \neq d(s) \in Z(L)$. Also, $s, d(s) \in Z(R)$ from the Lemma 2.2. Replacing p by $2ps$ in Equation (3.1) and using $char(R) \neq 2$, we get

$$[F(r), p]s + p[F(r), s] + F(p)[s, r] + [F(p), r]s + \alpha(p)[d(s), r] + [\alpha(p), r]d(s) \in Z(R).$$

Inthisexpression, using$s, d(s) \in Z(R)$ and Equation (3.1), we have

$$[\alpha(p), r]d(s) \in Z(R) \text{ for all } r, p \in L.$$

Hence, using$0 \neq d(s) \in Z(R)$ and Remark 2, we obtain

$$[\alpha(p), r] \in Z(R) \text{ for all } r, p \in L.$$

Replacing$r$ by $\alpha(r)$ in above relation, we have $[\alpha(p), \alpha(r)] \in Z(R)$ for all $r, p \in L$. Using the fact that α is automorphism, we get

$$\alpha([r, p]) \in Z(R) \text{ for all } r, p \in L.$$

FromtheRemark 1, weobtain

$$[r, p] \in Z(R) \text{ for all } r, p \in L.$$

Hence, fromtheLemma 2.3 weget$L \subseteq Z(R)$.

Theorem 3.2*Let ring R be prime with$char(R) \neq 2$, $(0) \neq L$ be a square closed Lie ideal of R and $0 \neq F: R \to R$ be a generalized $\alpha-$derivation associated with $\alpha-$derivation d such that $Z(L) \cap d\big(Z(L)\big) \neq (0)$ and $\alpha(L) \subset L$. If $F(r) \circ r \in Z(R)$ for all $r \in L$, then R is commutative.*

*Proof.*Let$F(r) \circ r \in Z(R)$ for all $x \in I$. Replacing r by $r + p$, we get

$$F(r) \circ p + F(p) \circ r \in Z(R) \text{ for all } r, p \in L. \tag{3.2}$$

From$Z(L) \cap d\big(Z(L)\big) \neq (0)$, we take fixed element $0 \neq s \in Z(L)$ which $0 \neq d(s) \in Z(L)$. Also, $s, d(s) \in Z(R)$ from the Lemma 2.2. Replacing p by $2ps$ in Equation (3.2) and using $char(R) \neq 2$, we have

$$(F(r) \circ p)s - p[F(r), s] + (F(p) \circ r)s + F(p)[s, r] + (\alpha(p) \circ r)d(s) + \alpha(p)[d(s), r] \in Z(R)$$

Inthisexpression, using$s, d(s) \in Z(R)$ and Equation (3.2), we get

$$(\alpha(p) \circ r)d(s) \in Z(R) \text{ for all } r, p \in L.$$

Hence, using$0 \neq d(s) \in Z(R)$ and Remark 2, we have

$$\alpha(p) \circ r \in Z(R) \text{ for all } r, p \in L.$$

Replacing$r$ by $\alpha(r)$ in above relation, we have $\alpha(p) \circ \alpha(r) \in Z(R)$ for all $r, p \in L$. Using the fact that α is automorphism, we obtain

$$\alpha(r \circ p) \in Z(R) \text{ for all } r, p \in L.$$

FromtheRemark 1, wehave

$$r \circ p \in Z(R) \text{ for all } r, p \in L.$$

Hence, fromtheLemma 2.4 weget$L \subseteq Z(R)$.

Lemma 3.3 *Let ring R be prime with$char(R) \neq 2$, $(0) \neq L$ be a square closedLie ideal of R and $0 \neq F: R \to R$ be a generalized $\alpha -$ derivation associated with $\alpha -$ derivation d such that $Z(L) \cap d(Z(L)) \neq (0)$ and $\alpha(L) \subset L$. If one of the following conditions is satisfy, then $L \subseteq Z(R)$.*

i. $[F(r), \alpha(p)] \in Z(R)$ for all $r, p \in L$.

ii. $F(r) \circ \alpha(p) \in Z(R)$ for all $r, p \in L$.

Proof.i. Let

$$[F(r), \alpha(p)] \in Z(R) \text{ for all } r, p \in L. \tag{3.3}$$

From$Z(L) \cap d(Z(L)) \neq (0)$, we take fixed element $0 \neq s \in Z(L)$ which $0 \neq d(s) \in Z(L)$. Replacing r by $2rs$ in Equation (3.3) and using $char(R) \neq 2$, we have

$$[F(r), \alpha(p)]s + F(r)[s, \alpha(p)] + \alpha(r)[d(s), \alpha(p)] + [\alpha(r), \alpha(p)]d(s) \in Z(R).$$

Inthisexpression, using$s, d(s) \in Z(R)$ from the Lemma 2.2 and Equation (3.3), we get

$$[\alpha(r), \alpha(p)]d(s) \in Z(R) \text{ for all } r, p \in L.$$

Hence, using$0 \neq d(s) \in Z(R)$ and Remark 2, we obtain

$$[\alpha(r), \alpha(p)] \in Z(R) \text{ for all } r, p \in L.$$

Using thefactthatα is automorphism, we get $\alpha([r, p]) \in Z(R)$ for all $x, y \in L$. From the Remark 1, we obtain $[r, p] \in Z(R)$ for all $r, p \in L$. Hence, from the Lemma 2.3 we get $L \subseteq Z(R)$.

ii. Let

$$F(r) \circ \alpha(p) \in Z(R) \text{ for all } r, p \in L. \tag{3.4}$$

From$Z(L) \cap d(Z(L)) \neq (0)$, we take fixed element $0 \neq s \in Z(L)$ which $0 \neq d(s) \in Z(L)$. Replacing r by $2rs$ in Equation (3.4) and using $char(R) \neq 2$, we have

$$(F(r) \circ \alpha(p))s + F(r)[s, \alpha(p)] + \alpha(r)[d(s), \alpha(p)] + (\alpha(r) \circ \alpha(p))d(s) \in Z(R).$$

Inthisexpression, using$s, d(s) \in Z(R)$ from the Lemma 2.2 and Equation (3.4), we have

$$(\alpha(r) \circ \alpha(p))d(s) \in Z(R) \text{ for all } r, p \in L.$$

Hence, using$0 \neq d(s) \in Z(R)$ and Remark 2, we obtain

$$\alpha(r) \circ \alpha(p) \in Z(R) \text{ for all } r, p \in L.$$

Using thefactthatα is automorphism, we get $\alpha(r \circ p) \in Z(R)$ for all $r, y \in L$. From the Remark 1, we obtain $r \circ p \in Z(R)$ for all $r, p \in L$. Hence, from the Lemma 2.3 we get $L \subseteq Z(R)$.

Theorem 3.4 *Let ring R be prime withchar$(R) \neq 2$, $(0) \neq L$ be a square closed Lie ideal of R and $0 \neq F: R \to R$ be a generalized $\alpha -$ derivation associated with $\alpha -$ derivation d such that $Z(L) \cap d\big(Z(L)\big) \neq (0)$ and $\alpha(L) \subset L$. If one of the following conditions is satisfy, then $L \subseteq Z(R)$.*

i. $[F(r), F(p)] - [r, p] \in Z(R)$ for all $r, p \in L$.

ii. $F(r) \circ F(p) - r \circ p \in Z(R)$ for all $r, p \in L$.

iii. $[F(r), F(p)] - r \circ p \in Z(R)$ for all $r, p \in L$.

iv. $F(r) \circ F(p) - [r, p] \in Z(R)$ for all $r, p \in L$.

Proof.i. By assumption,

$$[F(r), F(p)] - [r, p] \in Z(R) \text{ for all } r, p \in L. \tag{3.5}$$

From$Z(L) \cap d\big(Z(L)\big) \neq (0)$, we take fixed element $0 \neq s \in Z(L)$ which $0 \neq d(s) \in Z(L)$. Also, $s, d(s) \in Z(R)$ from the Lemma 2.2. Replacing p by $2ps$ in Equation (3.5) and using $char(R) \neq 2$, we have

$$[F(r), F(p)]s + [F(r), \alpha(p)]d(s) - [r, p]s \in Z(R).$$

Using Equation (3.5), weobtain

$$[F(r), \alpha(p)]d(s) \in Z(R) \text{ for all } r, p \in L.$$

FromRemark 2, weget

$$[F(r), \alpha(p)] \in Z(R) \text{ for all } r, p \in L.$$

So, $L \subseteq Z(R)$ from Lemma 3.3.

ii. By assumption,

$$F(r) \circ F(p) - r \circ p \in Z(R) \text{ for all } r, p \in L. \tag{3.6}$$

Since $Z(L) \cap d\big(Z(L)\big) \neq (0)$, we take fixed element $0 \neq s \in Z(L)$ which $0 \neq d(s) \in Z(L)$. Also, $s, d(s) \in Z(R)$ from the Lemma 2.2. Replacing p by $2ps$ in Equation (3.6) and using $char(R) \neq 2$, we obtain

$$\big(F(r) \circ F(p)\big)s + \big(F(r) \circ \alpha(p)\big)d(s) - (r \circ p)s \in Z(R).$$

Using Equation (3.6), weget

$$\big(F(r) \circ \alpha(p)\big)d(s) \in Z(R) \text{ for all } r, p \in L.$$

FromRemark 2, wehave

$$F(r) \circ \alpha(p) \in Z(R) \text{ for all } r, p \in L.$$

So, $L \subseteq Z(R)$ from Lemma 3.3.

iii. By assumption,

$$[F(r), F(p)] - r \circ p \in Z(R) \text{ for all } r, p \in L. \quad (3.7)$$

From $Z(L) \cap d(Z(L)) \neq (0)$, we take fixed element $0 \neq s \in Z(L)$ which $0 \neq d(s) \in Z(L)$. Also, $s, d(s) \in Z(R)$ from the Lemma 2.2. Replacing p by $2ps$ in Equation (3.7) and using $char(R) \neq 2$, we have

$$[F(r), F(p)]s + [F(r), \alpha(p)]d(s) - (r \circ p)s \in Z(R).$$

Using Equation (3.7), weobtain

$$[F(r), \alpha(p)]d(s) \in Z(R) \text{ for all } r, p \in L.$$

FromRemark 2, weget

$$[F(r), \alpha(p)] \in Z(R) \text{ for all } r, p \in L.$$

So, $L \subseteq Z(R)$ from Lemma 3.3.

iv. By assumption,

$$F(r) \circ F(p) - [r, p] \in Z(R) \text{ for all } r, p \in L. \quad (3.8)$$

From $Z(L) \cap d(Z(L)) \neq (0)$, we take fixed element $0 \neq s \in Z(L)$ which $0 \neq d(s) \in Z(L)$. Also, $s, d(s) \in Z(R)$ from the Lemma 2.2. Replacing p by $2ps$ in Equation (3.8) and using $char(R) \neq 2$, we obtain

$$(F(r) \circ F(p))s + (F(r) \circ \alpha(p))d(s) - [r, p]s \in Z(R).$$

Using Equation (3.8), weget

$$(F(r) \circ \alpha(p))d(s) \in Z(R) \text{ for all } r, p \in L.$$

FromRemark 2, wehave

$$F(r) \circ \alpha(p) \in Z(R) \text{ for all } r, p \in L.$$

So, $L \subseteq Z(R)$ from Lemma 3.3.

Theorem 3.5*Let ring R be prime with $char(R) \neq 2$, $(0) \neq L$ be a square closed Lie ideal of R and $0 \neq F: R \to R$ be a generalized $\alpha -$ derivation associated with $\alpha -$ derivation d such that $Z(L) \cap d(Z(L)) \neq (0)$ and $\alpha(L) \subset L$. If one of the following conditions is satisfy, then $L \subseteq Z(R)$.*

i. $F[r, p] - r \circ p \in Z(R)$ for all $r, p \in L$.

ii. $F(r \circ p) - [r, p] \in Z(R)$ for all $r, p \in L$.

iii. $[F(r), \alpha(p)] - [r, p] \in Z(R)$ for all $r, p \in L$.

iv. $F(r) \circ \alpha(p) - r \circ p \in Z(R)$ for all $r, p \in L$.

Proof.i. Let

$$F[r, p] - r \circ p \in Z(R) \text{ for all } r, p \in L. \quad (3.9)$$

From $Z(L) \cap d(Z(L)) \neq (0)$, we take fixed element $0 \neq s \in Z(L)$ which $0 \neq d(s) \in Z(L)$. Also, $s, d(s) \in Z(R)$ from the Lemma 2.2. Replacing p by $2ps$ in Equation (3.9) and using $char(R) \neq 2$, we have

$$F([r, p])s + \alpha([r, p])d(s) - (r \circ p)s + p[r, s] \in Z(R).$$

Inthisexpression, using $s \in Z(R)$ and Equation (3.9), we have

$$\alpha([r,p])d(s) \in Z(R) \text{ for all } r,p \in L.$$

Hence, using $0 \neq d(s) \in Z(R)$ and Remark 2, we get

$$\alpha([r,p]) \in Z(R) \text{ for all } r,p \in L.$$

FromtheRemark 1, weobtain $[r,p] \in Z(R)$ for all $r,p \in L$. Hence, from the Lemma 2.3 we get $L \subseteq Z(R)$.

ii. Let

$$F(r \circ p) - [r,p] \in Z(R) \text{ for all } r,p \in L. \tag{3.10}$$

From $Z(L) \cap d\big(Z(L)\big) \neq (0)$, we take fixed element $0 \neq s \in Z(L)$ which $0 \neq d(s) \in Z(L)$. Also, $s, d(s) \in Z(R)$ from the Lemma 2.2. Replacing p by $2ps$ in Equation (3.10) and using $char(R) \neq 2$, we get

$$F(r \circ p)s + \alpha(r \circ p)d(s) - [r,p]s \in Z(R).$$

Using Equation (3.4), we obtain

$$\alpha(r \circ p)d(s) \in Z(R) \text{ for all } r,p \in L.$$

Hence, using $0 \neq d(s) \in Z(R)$ and Remark 2, we obtain

$$\alpha(r \circ p) \in Z(R) \text{ for all } r,p \in L.$$

FromtheRemark 1, wehave $r \circ p \in Z(R)$ for all $r, y \in L$. Hence, from the Lemma 2.3 we get $L \subseteq Z(R)$.

iii. Let

$$[F(r),\alpha(p)] - [r,p] \in Z(R) \text{ for all } r,p \in L. \tag{3.11}$$

From $Z(L) \cap d\big(Z(L)\big) \neq (0)$, we take fixed element $0 \neq s \in Z(L)$ which $0 \neq d(s) \in Z(L)$. Also, $s, d(s) \in Z(R)$ from the Lemma 2.2. Replacing r by $2rs$ in Equation (3.11) and using $char(R) \neq 2$, we have

$$[F(r),\alpha(p)]s + [\alpha(r),\alpha(p)]d(s) - [r,p]s \in Z(R).$$

FromEquation (3.11), we obtain

$$[\alpha(r),\alpha(p)]d(s) \in Z(R) \text{ for all } r,p \in L.$$

Hence, using $0 \neq d(s) \in Z(R)$ and Remark 2, we get

$$[\alpha(r),\alpha(p)] \in Z(R) \text{ for all } r,p \in L.$$

Using thefactthat α is automorphism, we have $\alpha([r,p]) \in Z(R)$.From the Remark 1, we obtain $[r,p] \in Z(R)$ for all $r,p \in L$. Hence, from the Lemma 2.3 we get $L \subseteq Z(R)$.

iv. Let

$$F(r) \circ \alpha(p) - r \circ p \in Z(R) \text{ for all } r,p \in L. \tag{3.12}$$

From $Z(L) \cap d\big(Z(L)\big) \neq (0)$, we take fixed element $0 \neq s \in Z(L)$ which $0 \neq d(s) \in Z(L)$. Also, $s, d(s) \in Z(R)$ from the Lemma 2.2. Replacing r by $2rs$ in Equation (3.12) and using $char(R) \neq 2$, we obtain

$$\big(F(r) \circ \alpha(p)\big)s + \big(\alpha(r) \circ \alpha(p)\big)d(s) - (r \circ p)s \in Z(R).$$

Using Equation (3.12), we have

$$\big(\alpha(r) \circ \alpha(p)\big)d(s) \in Z(R) \text{ for all } r,p \in L.$$

Hence, using $0 \neq d(s) \in Z(R)$ and Remark 2, we get

$$\alpha(r) \circ \alpha(p) \in Z(R) \text{ for all } r, p \in L.$$

Using thefactthat α is automorphism, we get $\alpha(r \circ p) \in Z(R)$.From the Remark 1, we obtain $r \circ p \in Z(R)$ for all $r, p \in L$. Hence, from the Lemma 2.3 we get $L \subseteq Z(R)$.

Theorem 3.6 *Let ring R be prime with* $char(R) \neq 2$, $(0) \neq L$ *be a square closed Lie ideal of* R *and* $0 \neq F, G: R \to R$ *are generalized* $\alpha -$ *derivations associated with* $\alpha -$ *derivations* d *and* g *respectively such that* $Z(L) \cap d\big(Z(L)\big) \neq (0), Z(L) \cap g\big(Z(L)\big) \neq (0)$ *and* $\alpha(L) \subset L$. *If one of the following conditions is satisfy, then* $L \subseteq Z(R)$.

i. $[F(r), G(p)] - [r, p] \in Z(R)$ for all $r, p \in L$.

ii. $F[r, p] - [p, G(r)] \in Z(R)$ for all $r, p \in L$.

iii. $F(r \circ p) - p \circ G(r) \in Z(R)$ for all $r, p \in L$.

Proof.i) For all $r, p \in L$, let

$$[F(r), G(p)] - [r, p] \in Z(R). \tag{3.13}$$

Byhypothesis, $Z(L) \cap g\big(Z(L)\big) \neq (0)$. Then, we take fixed element $0 \neq s \in Z(L)$ which $0 \neq g(s) \in Z(L)$. Also, $s, g(s) \in Z(R)$ from the Lemma 2.2. Replacing p by $2ps$ in Equation (3.13) and using $char(R) \neq 2$, we get

$$[F(r), G(p)]s + [F(r), \alpha(p)]g(s) - [r, p]s \in Z(R) \text{ for all } r, p \in L.$$

FromEquation (3.13), we obtain

$$[F(r), \alpha(p)]g(s) \in Z(R) \text{ for all } r, p \in L.$$

Inthisexpression, using $0 \neq g(s) \in Z(R)$ and Remark 2, we have

$$[F(r), \alpha(p)] \in Z(R) \text{ for all } r, p \in L.$$

FromtheLemma 3.3, weget $L \subseteq Z(R)$.

ii) For all $r, p \in L$, let

$$F[r, p] - [p, G(r)] \in Z(R). \tag{3.14}$$

Byhypothesis, $Z(L) \cap d\big(Z(L)\big) \neq (0)$. Then, we take fixed element $0 \neq s \in Z(L)$ which $0 \neq d(s) \in Z(L)$. Also,s, $d(s) \in Z(R)$ from the Lemma 2.2. Replacing p by $2ps$ in Equation (3.14) and using $char(R) \neq 2$, we get

$$F([r, p])s + \alpha([r, p])d(s) - [p, G(r)]s \in Z(R) \text{ for all } r, p \in L.$$

FromEquation (3.14), we obtain

$$\alpha([r, p])d(s) \in Z(R) \text{ for all } r, p \in L.$$

Inthisexpression, using $0 \neq d(s) \in Z(R)$ and Remark 2, we get

$$\alpha([r, p]) \in Z(R) \text{ for all } r, p \in L.$$

FromtheRemark 1, weobtain $[r, p] \in Z(R)$ for all $r, p \in L$. Hence, from the Lemma 2.3 we get $L \subseteq Z(R)$.

iii) For all $r, y \in I$, let

$$F(r \circ p) - p \circ G(r) \in Z(R). \tag{3.15}$$

Byhypothesis, $Z(L) \cap d\big(Z(L)\big) \neq (0)$. Then, we take fixed element $0 \neq s \in Z(L)$ which $0 \neq d(s) \in Z(L)$. Also, $s, d(s) \in Z(R)$ from the Lemma 2.2. Replacing p by $2ps$ in Equation (3.15) and using $char(R) \neq 2$, we have

$$F(r \circ p)s + \alpha(r \circ p)d(s) - \big(p \circ G(r)\big)s \in Z(R) \text{ for all } r, p \in L.$$

FromEquation (3.15), we get

$$\alpha(r \circ p)d(s) \in Z(R) \text{ for all } r, p \in L.$$

Inthisexpression, using$0 \neq d(s) \in Z(R)$ and Remark 2, we have

$$\alpha(r \circ p) \in Z(R) \text{ for all } r, p \in L.$$

FromtheRemark 1, weobtain$r \circ p \in Z(R)$ for all $r, p \in L$. Hence, from the Lemma 2.3 we get $L \subseteq Z(R)$.

Theorem 3.7 *Let ring R be prime with$char(R) \neq 2$, $(0) \neq L$ be a square closed Lie ideal of R and $0 \neq F, G: R \to R$ are generalized $\alpha -$ derivations associated with $\alpha -$ derivations d and g respectively such that $\{s \in Z(L) | 0 \neq d(s), 0 \neq g(s) \in Z(L), d(s) \neq \mp g(s)\} \neq \emptyset$ and $\alpha(L) \subset L$. If one of the following conditions is satisfy, then $L \subseteq Z(R)$.*

i. $[F(r), r] - [r, G(r)] \in Z(R)$ for all $r \in L$.

ii. $F(r) \circ r - r \circ G(r) \in Z(R)$ for all $r \in L$.

Proof.i. Let $[F(r), r] - [r, G(r)] \in Z(R)$ for all $r \in I$. Replacing r by $r + p$ for any $p \in L$, we get

$$[F(r), p] + [F(p), r] - [r, G(p)] - [p, G(r)] \in Z(R) \text{ for all } r, p \in L \tag{3.16}$$

From$\{s \in Z(L) | 0 \neq d(s), 0 \neq g(s) \in Z(L)\} \neq \emptyset$, we take fixed element $0 \neq s \in Z(L)$ which $0 \neq d(s), 0 \neq g(s) \in Z(L) \subset Z(R)$. Replacing p by $2ps$ in Equation (3.16) and using $char(R) \neq 2$, we get

$$[F(r), p]s + [F(p), r]s + [\alpha(p), r]d(s) - s - [r, \alpha(p)]g(s) - [p, G(r)]s \in Z(R)$$

FromEquation (3.16), we have

$$[\alpha(p), r]d(s) - [r, \alpha(p)]g(s) \in Z(R) \text{ for all } r, p \in L$$

Using equation$-[r, \alpha(p)] = [\alpha(p), r]$, we get

$$[\alpha(p), r]\big(d(s) + g(s)\big) \in Z(R) \text{ for all } r, p \in L$$

Hence, using$0 \neq d(s), 0 \neq g(s) \in Z(R)$ and Remark 2, we obtain

$$[\alpha(p), r] \in Z(R) \text{ for all } r, p \in L.$$

Using sameprocess in Theorem 3.1, weget$R$ is commutative.

ii. Let $F(r) \circ r - r \circ G(r) \in Z(R)$ for all $r \in L$. Replacing r by $r + p$ for any $p \in L$, we get

$$F(r) \circ p + F(p) \circ r - r \circ G(p) - p \circ G(r) \in Z(R) \text{ for all } r, p \in L \tag{3.17}$$

From$\{s \in Z(L) | 0 \neq d(s), 0 \neq g(s) \in Z(L)\} \neq \emptyset$, we take fixed element $0 \neq s \in Z(L)$ which $0 \neq d(s), 0 \neq g(s) \in Z(L) \subset Z(R)$. Replacing p by $2ps$ in Equation (3.17) and using $char(R) \neq 2$, we get

$$(F(r) \circ p)s + (F(p) \circ r)s + (\alpha(p) \circ r)d(s) - \big(r \circ G(p)\big)s - \big(r \circ \alpha(p)\big)g(s) - \big(p \circ G(r)\big)s \in Z(R)$$

FromEquation (3.17), we have

$$(\alpha(p) \circ r)d(s) - \big(r \circ \alpha(p)\big)g(s) \in Z(R) \text{ for all } r, p \in L$$

Using equation$r \circ \alpha(p) = \alpha(p) \circ r$, we get

$$(\alpha(p) \circ r)\big(d(s) - g(s)\big) \in Z(R) \text{ for all } r, p \in L$$

Hence, using$0 \neq d(s), 0 \neq g(s) \in Z(R)$ and Remark 2, we obtain

$$\alpha(p) \circ r \in Z(R) \text{ for all } r, p \in L.$$

Using sameprocess in Theorem 3.1, weget$R$ is commutative.

REFERENCES

[1] N.Argaç, "On near-rings with two-sided α -derivations", *Turk. J.Math*, vol. 28, pp. 195–204, 2004.

[2] O. Atay, N. Aydınand B. Albayrak, "Generalized reverse derivations on closed Lie ideals", *Journal of Scientific Perspectives*, vol. 2(3), pp. 61-74, 2018.

[3] M. Bresar,"On thedistance of thecomposition of twoderivationstothegeneralizedderivations", *Glaskow Math. J.*, vol. 33, pp. 89-93, 1991.

[4] J. C. Chang, "Right generalized$(\alpha, \beta) -$ derivations having power central values", *Taiwanese J. Math*.,vol. 13(4), pp. 1111-1120, 2009.

[5] N. Rehman, "On commutativity of Rings withGeneralizedDerivations", *Math. J. OkayamaUniv.*, vol.44, pp. 43-49, 2002.

[6] N. Rehman, H. Motoshiand R. M. Al-Omary,"CentralizingMappings, MoritaContextandGeneralized$(\alpha, \beta) -$derivations", *J.TaibahUniv. Sci.*,vol.8(4), pp. 370-374, 2014.