Double-Framed G-Invariant N-Fuzzy Soft G-Modules

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Abstract: *The target of this study is to observe some of the algebraic structures of a double-framed N- fuzzy soft set. So, we introduce the concept of a double-framed N-fuzzy soft G-module of a given classical module and investigate some of the crucial properties and characterizations of the proposed concept. The ideas of G-invariant double-framed N- fuzzy soft G-modules are also discussed.*

Keywords: *Group action, classical G-module, fuzzy set, soft set, double-framed N- fuzzy soft G-module, G-invariant, Cartesian product and intersection.*

I. INTRODUCTION

L.A. Zadeh [12] in 1965 introduced the concept of fuzzy sets to describe vagueness mathematically in its very abstractness. The theory of G-modules originated in the 20th century. Representation theory was developed on the basis of embedding a group G in to a linear group GL(V).The theory of group representation (G module theory) was developed by Frobenious. G in 1962. Soon after the concept of fuzzy sets were introduced by Zadeh in 1965. Fuzzy subgroup and its important properties were defined and established by Rosenfeld in 1971. After that in the year 2004, Shery Fernandez [10] introduced fuzzy parallels of the notions of G-modules. The concept of group actions in various algebraic structures in [5, 9]. Jun et.al [6] initiated to introduce double framed soft sets and presented its applications in BCK/BCI algebras. Let X be a non-empty set. A mapping $\mu: X \to [0, 1]$ is called a fuzzy subset of X. Rosenfeld [8] applied the concept of fuzzy sets to the theory of groups and defined the concept of fuzzy subgroups of a group. Since then, many papers concerning various fuzzy algebraic structures have appeared in the literature [1, 2-4, 7, 11]. Jun et.al [6] studied double-framed soft sets in BCK/BCI algebras.The target of this study is to observe some of the algebraic structures of a double-framed N-fuzzy soft set. So, we introduce the concept of a double-framed N-fuzzy soft G-module of a given classical module and investigate some of the crucial properties and characterizations of the proposed concept. The ideas of G-invariant double-framed N- fuzzy soft G-modules are also discussed.

II. PRELIMINARIES

Let 'M' be a module over the ring of integers Z and G be a finite group which acts on M ((ie) $\forall\ g \in G,\ x \in M,\ x^g = gxg^{-1} \in M$). The identity element of G is denoted by "e".

Definition 2.1: Let G be a finite group. A vector space M over a field K (a subfield of C) is called a G-module if for every g ∈ G and m ∈M, there exists a product (called the right action of G on M) m.g∈ M which satisfies the following axioms.

1. m.1G = m for all m ∈ M (1G being the identify of G)
2. m. (g. h) = (m.g). h, m ∈ M, g, h ∈ G
3. $(k_1 m_1 + k_2 m_2).\ g = k_1 (m_1.\ g) + k_2(m_2.\ g)$, $k_1, k_2 \in K$, $m_1, m_2 \in M$ & g ∈ G.

In a similar manner ,the left action of G on M can be defined.

***Definition 2.2*:**Let M and M* be G-modules. A mapping Ø: M→M* is a G-module homomorphism if

1. $Ø(k_1 m_1 + k_2 m_2) = k_1\, Ø(m_1) + k_2\, Ø(m_2)$
2. $Ø(gm) = g\, Ø(m)$, $k_1, k_2 \in K$, $m, m_1, m_2 \in M$ & $g \in G$.

***Definition 2.3*:**Let M be a G-module. A subspace N of M is a G - sub module if N is also a G-module under the action of G.

Definition 2.4: Let U be any universal set, E set of parameters and $A \subseteq E$. Then a pair (K,A) is called soft set over U, where K is a mapping from A to 2^U, the power set of U.

Example 2.5: Let $X=\{c_1,c_2,c_3\}$ be the set of three cars and E={costly(e_1), metallic colour(e_2), cheap(e_3)}be the set of parameters, where $A=\{e_1,e_2\} \subset E$. Then $(K,A)=\{K(e_1)=\{c_1,c_2,c_3\},K(e_2)=\{c_1,c_2\}\}$ is the crisp soft set over X.

Definition 2.6: Let U be an initial universe. Let P (U) be the power set of U, E be the set of all parameters and $A\subseteq E$. A soft set (f_A, E) on the universe U is defined by the set of order pairs (f_A, E) = {(e, f_A (e)): e∈ E, f_A∈ P (U)} where f_A : E → P (U) such that f_A (e) = ϕ, if e ∉A. Here 'f_A' is called an approximate function of the soft set.

Example 2.7: Let U = $\{u_1,u_2, u_3, u_4\}$ be a set of four shirts and E = {white(e_1),red(e_2), blue(e_3)} be a set of parameters. If A = $\{e_1, e_2\} \subseteq$ E. Let $f_A(e_1) = \{u_1, u_2 ,u_3, u_4\}$ and $f_A(e_2)= \{u_1,u_2,u_3\}$. Then we write the soft set (f_A, E)= {(e_1, $\{u_1, u_2 ,u_3, u_4\}$), (e_2,$\{u_1,u_2,u_3\}$)} over U which describe the "colour of the shirts" which Mr. X is going to buy. We may represent the soft set in the following form: U={(e_1,u_1), (e_2,u_1), (e_1,u_2), (e_2,u_2), (e_1,u_3), (e_2,u_3), (e_1, u_4)}.

Definition 2.8: Let U be the universal set, E set of parameters and $A\subset E$. Let K(X) denote the set of all fuzzy subsets of U. Then a pair (K,A) is called fuzzy soft set over U, where K is a mapping from A to K(U).

Example 2.9: Let $U=\{c_1,c_2,c_3\}$ be the set of three cars and E={costly(e_1), metallic colour(e_2), cheap(e_3)} be the set of parameters, where $A=\{e_1,e_2\}\subset E$. Then $(K,A)=\{K(e_1)=\{c_1/0.6,c_2/0.4,c_3/0.3\}, K(e_2)=\{c_1/0.5,c_2/0.7,c_3/0.8\}\}$ is the fuzzy soft set over U denoted by F_A.

Definition 2.10: Let K_A be a fuzzy soft set over U and 'α' be a subset of U. Then upper α- inclusion of K_A denoted by $K^{\alpha}_A = \{x\in A / K(x) \geq \alpha\}$. Similarly $K^{\alpha}_A = \{x\in A / K(x) \leq \alpha\}$ is called lower α-inclusion of K_A.

Definition 2.11: Let K_A and G_B be fuzzy soft sets over the common universe U and ψ:A→ B be a function. Then fuzzy soft image of K_A under ψ over U denoted by ψ(K_A) is a set-valued function, where ψ(K_A):B→ 2^U defined by ψ(K_A) (b)={∪{K(a) / a∈A and ψ (a)=b}, if ψ^{-1}(b)≠ φ} for all b∈B, the soft pre-image of G_B under ψ over U denoted by $\psi^{-1}(G_B)$ is a set-valued function, where$\psi^{-1}(G_B)$: A → 2^U defined by $\psi^{-1}(G_B)(b)$ = G(ψ(a)) for all a ∈A. Then fuzzy soft anti-image of K_A under ψ over U denoted by ψ(K_A) is a set-valued function, where ψ(K_A):B → 2^U defined by$\psi^{-1}(K_A)$(b)={∩{K(a) / a∈A and ψ (a)=b} , if ψ^{-1}(b) ≠ φ} for all b∈B.

Definition 2.12: A group action of G on a fuzzy soft set 'A' of a Z-module M is denoted by A^g and is defined by $A^g(x)=A(x^g), g\in G.$

From the definition of group action G on a fuzzy soft set, following results are easy to verify.

Lemma 2.13 [P.K.Sharma]: Let G be a finite group which acts on Z-module M. Then for every $x, y\in M,\ g\in G \ and\ r\in Z,$ we have

(i) $(x+y)^g = x^g + y^g$

(ii) $(xy)^g = x^g y^g$

(iii) $(rx)^g = r\,x^g$

(iv) $(x,y)^g = (x^g, y^g)$

Proof: (i) Since $(x+y)^g = g(x+y)g^{-1} = gxg^{-1} + gyg^{-1} = = x^g + y^g$

(ii) $(xy)^g = g(xy)g^{-1} = g(xey)g^{-1} = g(xg^{-1}gy)g^{-1} = (gxg^{-1})(gyg^{-1}) = x^g y^g$

(iii) $(rx)^g = g(rx)g^{-1} = g(x+x+........+r\,times)g^{-1}$

$= gxg^{-1} + gxg^{-1} + gxg^{-1} + ...+ r\,times$

$= r(gxg^{-1}) = r\ x^g$

(iv) $(x,y)^g = g(x,y)g^{-1} = (gxg^{-1}, gyg^{-1}) = (x^g, y^g)$

III. DOUBLE-FRAMED N- FUZZY SOFT G-MODULES

In this section, we define the concept of double-framed N- fuzzy soft G-module of a given classical module over a ring and also investigate its elementary properties and characterizations. Throughout this paper, R denotes a commutative ring with unity 1.

***Definition 3.1*:** A negative fuzzy (N- fuzzy) soft set $\mathcal{A}$ on the universe of discourse $\mathcal{X}$ is defined as $\mathcal{A} = \langle x, \bar{\delta}_{\mathcal{A}}(x)\rangle$, $x \in \mathcal{X}$,where $\delta : \mathcal{X} \rightarrow [-1, 0]$ and $-1 \leq \bar{\delta}_{\mathcal{A}}(x) \leq 0$.

***Definition 3.2*:**A double-framed N- fuzzy set (DFNFS) A on a universe X is an object of the form A={(x,μ_A(x),ν_A(x))|x ∈ X}where μ_A(x) ∈[-1,0] is called the degree of truth membership(TM) of x in A, ν_A(x) ∈ [-1,0] is called the degree of false membership (FM) of x in A. μ_A (x),ν_A(x) must satisfy the condition μ_A (x)+ν_A(x) ≤1∀x ∈ X. Then ∀x ∈ X, 1−(μ_A (x)+ν_A(x)) is called the degree of refusal membership of x in A.

***Example 3.3*:** Ms. A wants to buy a dress and looks at the shops all day to find the most beautiful one which supplies her criterions. Since she is not sure for the parameters, she may give up some of them if she finds her cup of tea and also she does not spare much time for looking around. U = The set of all dresses under consideration in the shop. $U = \{x_1, x_2, x_3, x_4\}$. E =The set of all parameters. Each parameter is a word in sentence. $E = \{e_1, e_2, e_3, e_4, e_5\}$. e_1:=low, e_2:=high, e_3:=colour, e_4:=plain and e_5:=design .The double framed soft fuzzy set is defined as follows. $(F, f) = \{(F(e_1), f(e_1)), (F(e_2), f(e_2)), (F(e_3), f(e_3)), (F(e_4), f(e_4)), (F(e_5), f(e_5))\} = \{(\{x_4\}, 0.6), (\{x_3, x_4\}, 1), (\{x_1, x_2\}, 0.3), (\{x_1, x_3\}, 0.8), (\{x_4\}, 0.7)\}$ .Here the values $f(e_i)$ indicates the influences of the parameters on the decision in what degree.

Definition 3.4: Let 'M' be a module over a ring R. A double-framed N- fuzzy soft set 'A' on M is called a double-framed N- fuzzy soft G-module of M if the following conditions are satisfied:

(DFNFSGM–1) : $A(0) = X$

(ie) $P_A(0)=1, \quad I_A(0)=0$

(DFNFSGM–2) : $A(x+y) \geq \min\{A(x), A(y)\}, \ \textit{for each} \ \ x, y \in M$

(ie) $P_A(x+y) \geq \min\{P_A(x), P_A(y)\},$

$I_A(x+y) \leq \max\{I_A(x), I_A(y)\}.$

(DFNFSGM–3) : $A(rx) \geq A(x), \ \ \textit{for each} \ x \in M, \ r \in R$

(ie) $P_A(rx) \geq P_A(x), \ I_A(rx) \leq I_A(x).$

The collection of all double-framed N- fuzzy soft G-modules of M is denoted by DFNFSG(M).

Example 3.5: Let us take the classical ring R = Z_4 = {0, 1, 2, 3}. Since each ring is a module on itself, we consider M = Z_4 as a classical module.

Define a double-framed N-fuzzy soft set 'A' as follows:

$$A=\{(1, 1, 0)/0+(0.7, 0.4, 0.7)/1+(0.8, 0.2, 0.4)/2+(0.7, 0.2, 0.7)/3\}.$$

It is clear that the double-framed N- fuzzy soft set 'A' is a double-framed N- fuzzy soft G-module of M.

Proposition 3.6: Let A be a double-framed N- fuzzy soft set of Z-module M and G be a finite group which acts on M. Then A^g is also a double-framed N-fuzzy soft G-module of M.

Proof: Clearly, (DFNFSGM–1): $A^g(0) = A(0^g) = A(0) = 1.$

(DFNFSGM–2): Let $x, y \in M, \ g \in G \ and \ r \in Z,$ then by Lemma 2.13 (i),

$$A^g(x+y) = A\{(x+y)^g\} = A(x^g + y^g) \geq \min\{A(x^g), A(y^g)\} = \min\{A^g(x), \ A^g(y)\}.$$

(DFNFSGM–3): $A^g(rx) = A\{(rx)^g\} = A(r\, x^g) \geq A(x^g) = A^g(x)$ by Lemma 2.13 (i) and (iii).

Hence, A^g is double-framed N- fuzzy soft G-module of M.

Remark 3.7: The converse of proposition 3.6 does not hold.

Example 3.8: Let $M = \{Z_4 = \{0, 1, 2, 3\}, +4, X_4\}$ regarded as Z-module and a finite group $G = (\{0, 1, 2, 3, 4\}, X_5)$.consider a double-framed N- fuzzy soft set A of M given by $A(0) = 0.2, A(1) = 0.3, A(2) = 0.7, A(3) = 0.4.$ Clearly 'A' is not double-framed N- fuzzy soft G-module of M, because

$$A(2+4^3) = A(1) = 0.3 < 0.4 = \min\{0.7,\ 0.4\} = \min\{A(2),\ A(3)\}.$$

Take g = 3, so that $g^{-1} = 2$, then

$$x^g = gxg^{-1} = 3\,X_4\ x\ X_4\ 2 = 6x(\text{mod}\,4) = 2x(\text{mod}\,4),$$

We get $A^g(x) = A(x^g) = \begin{cases} 1, & if\ \ x = 0,\ 2 \\ 0.4, & if\ \ x = 1,\ 3 \end{cases}$

Now, it is easy to check that A^g is double-framed N- fuzzy soft G-module of M.

Definition 3.9: Let A and B be double-framed N- fuzzy soft sets on M. Then their sum A+B is a double-framed N-fuzzy soft set on M, defined as follows:

$$P_{A+B}(x) = \max\{\min\{P_A(y),\ P_B(z)\}/\ x = y+z,\ y,z \in M\}$$
$$I_{A+B}(x) = \min\{\max\{I_A(y),\ I_B(z)\}/\ x = y+z,\ y,z \in M\}$$

Definition 3.10: Let A be a double-framed N- fuzzy soft set on M. Then A is a double-framed N- fuzzy soft set on M, defined as follows:

$$P_{-A}(x) = P_A(-x),\ \ I_{-A}(x) = I_A(-x),\ \ for\ each\ \ x \in M.$$

IV. CHARACTERIZATION OF DOUBLE-FRAMED N- FUZZY SOFT G-MODULES

In this section, we will discuss the structure of double-framed N- fuzzy soft G-modules under group actions. The following theorems are proved enhanced with group actions.

Definition 4.1: Let A be a double-framed N- fuzzy soft set on M and $r \in R$. Define double-framed N- fuzzy soft set rA on M as follows:

$$P_{rA} = \max\{P_A(y)/\ y \in M,\ \ x = ry\}$$
$$I_{rA} = \min\{I_A(y)/\ y \in M,\ \ x = ry\}.$$

Proposition 4.2: If A is a double-framed N- fuzzy soft set of G-module of M and G be a finite group which acts on M, then (–1)A = – A.

Proof: Let $x \in M$ be arbitrary.

$$P_{(-1)A}(x^g) = \bigvee_{x=(-1)y} P_A(y^g) = \bigvee_{y=-x} P_A(x^g) = P_A(-x^g) = P_{-A}(x^g)$$

Then the following is valid.

$$(-1)A = (P_{(-1)A},\ \ I_{(-1)A}) = (P_{-A},\ I_{-A}) = -A$$

This completes the proof.

Proposition 4.3: If A and B are double-framed N- fuzzy soft sets on M with $A \subseteq B,\ then\ rA \subseteq rB,$ for each $r \in R$.

Proof: It is straightforward by the definition 4.1.

Proposition 4.4: If A and B are double-framed N- fuzzy soft sets on M and G be a finite group which acts on M, then $r(sA) = (rs)A,$ for each r, $s \in R$.

Proof: Let $x \in M$ and r, $s \in R$ be arbitrary.

$$I_{r(sA)}(x^g) = \bigwedge_{x=ry} I_{sA}(y^g) = \bigwedge_{x=ry}\ \bigwedge_{y=sx} I_A(z^g) = \bigwedge_{x=r(sz)} I_A(z^g) = \bigwedge_{x=(rs)z} I_A(z^g) = I_{(rs)A}(x^g)$$

By the similar calculations the other equality is obtained, so

$$r(sA) = (P_{r(sA)},\ \ I_{r(sA)}) = (P_{(rs)A},\ I_{(rs)A}) = (rs)A$$

Hence the proof.

Proposition 4.5: If A and B are double-framed N- fuzzy soft sets on M and G be a finite group which acts on M, then $r(A+B) = rA + rB$ for each $r \in R$.

Proof: Let A and B be double-framed N- fuzzy soft sets on M, $x \in M$ and $r \in R$.

$$N_{r(A+B)}(x^g) = \bigvee_{x=ry} N_{A+B}(y^g)$$

$$= \bigvee_{x=ry}\ \bigvee_{y=x_1+x_2} \min\{N_A(z_1^g),\ N_B(z_2^g)\}$$
$$= \bigvee_{x=rx_1+rx_2} \min\{N_A(z_1^g),\ N_B(z_2^g)\}$$
$$= \bigvee_{x=x_1+x_2} \min\left\{\left(\left(\bigvee_{x_1=rz_1} N_A(z_1^g)\right), \left(\bigvee_{x_2=rz_2} N_B(z_2^g)\right)\right)\right\}$$
$$= \bigvee_{x=x_1+x_2} \min\{N_{rA}(x_1^g),\ N_{rB}(x_2^g)\}$$
$$= N_{rA+rB}(x^g).$$

The other equality is obtained similarly.
Hence, $r(A+B) = (P_{r(A+B)},\ I_{r(A+B)}) = (P_{rA+rB},\ I_{rA+rB}) = rA+rB.$

Proposition 4.6: If A is double-framed N- fuzzy soft set on M and G be a finite group which acts on M, then $P_{rA}(rx)^g \geq P_A(x^g),\quad I_{rA}(rx)^g \leq I_A(x^g).$
Proof: It is straightforward by the definition 4.1.

Proposition 4.7: If A and B are double-framed N- fuzzy soft sets on M and G be a finite group which acts on M, then

(i) $P_B(rx)^g \geq P_A(x^g),\quad \textit{for each } x \in M, \textit{ if and only if } P_{rA} \leq P_B$

(ii) $I_B(rx)^g \leq I_A(x^g),\quad \textit{for each } x \in M, \textit{ if and only if } I_{rA} \geq I_B$

Proof: (i) Suppose $P_B(rx)^g \geq P_A(x^g),\quad \textit{for each } x \in M,$ then

$$P_{rA}(x^g) = \bigvee_{x=ry,\ y\in M} P_A(y^g),$$

So, $P_{rA} \leq P_B$.

Conversely, suppose $P_{rA} \leq P_B$ is satisfied, then $P_{rA}(x^g) \leq P_B(x^g),\quad \textit{for each } x \in M.$
Hence, $P_B(x^g) \geq P_{rA}(rx)^g \geq P_A(x^g),\quad \textit{for each } x \in M$ (by Proposition 4.6), (ii) is also proved in a similar way.

Proposition 4.8: If A and B are double-framed N- fuzzy soft sets on M and G be a finite group which acts on M, then

(i) $P_{rAsB}(rx+sy)^g \geq \min\{P_A(x^g),\ P_B(y^g)\}$

(ii) $I_{rAsB}(rx+sy)^g \leq \max\{I_A(x^g),\ I_B(y^g)\},\quad \textit{for each } x \in M,\ r,s \in R.$

Proof: It is proved by using Definition 3.4, Definition 3.10 and Proposition 4.6.

Proposition 4.9: If A, B and C are three double-framed N- fuzzy soft sets on M and G be a finite group which acts on M, then the following are satisfied for each r, s $\in$ R:

(i) $P_C(rx+sy)^g \geq \min\{P_A(x^g),\ P_B(y^g)\},\ \textit{for all } x,y \in M \textit{ if and only if } P_{rA+sB} \leq P_C$

(ii) $I_C(rx+sy)^g \leq \max\{I_A(x^g),\ I_B(y^g)\},\ \textit{for all } x,y \in M \textit{ if and only if } I_{rA+sB} \geq I_C$

Proof: It is proved by using Proposition 4.8.

Theorem 4.10: Let A be a double-framed N-fuzzy soft set on M and G be a finite group which acts on M and for each r, s $\in$ R, then

(i) $P_{rA} \leq P_A \iff P_A(rx)^g \geq P_A(x^g),$
$I_{rA} \geq I_A \iff I_A(rx)^g \leq I_A(x^g)\ \textit{for each } x \in M.$

(ii) $P_{rA+sA} \leq P_A \iff P_A(rx+sy)^g \geq \min\{P_A(x^g),\ P_A(y^g)\}$

$$I_{rA+sA} \geq I_A \iff I_A(rx+sy)^g \leq \max\{I_A(x^g), I_A(y^g)\}$$

Proof: The Proof follows from Proposition 4.7 and Proposition 4.9.

Theorem 4.11: Let A be a double-framed N- fuzzy soft set on M and G be a finite group which acts on M. Then A $\in$ DFNFSGM(M) if and only if the following properties are satisfied.

(i) $A(0^g) = \tilde{X}$.

(ii) $A(rx+sy)^g \geq \min\{A(x^g), A(y^g)\}$, *for each* $x, y \in M$ *and* $r, s \in R$.

Proof: Let A be a double-framed N- fuzzy soft G-module on M and G be a finite group which acts on M and x, y $\in$ M.

From the condition (DFNFSGM–1) of Definition 3.4, it is obvious that $A(0^g) = \tilde{X}$. From (DFNFSGM–2) and (DFNFSGM–3), the following are true,

$$P_A(rx+sy)^g \geq \min\{P_A(rx^g), P_A(sy^g)\} \geq \min\{P_A(x^g), P_A(y^g)\}$$

$$I_A(rx+sy)^g \leq \max\{I_A(rx)^g, I_A(sy)^g\}$$

$$\leq \max\{I_A(x^g), I_A(y^g)\} \text{ for each } x, y \in M \text{ and } r, s \in R.$$

Hence, $A(rx+sy)^g = (P_A(rx+sy)^g, N_A(rx+sy)^g, I_A(rx+sy)^g)$

$$\geq (\min\{P_A(x^g), P_A(y^g)\}, \min\{N_A(x^g), N_A(y^g)\}, \min\{I_A(x^g), I_A(y^g)\})$$

$$= (\min\{P_A(x^g), N_A(x^g), I_A(x^g)\}, \min\{P_A(y^g), N_A(y^g), I_A(y^g)\})$$

$$= \min\{A(x^g), A(y^g)\}.$$

Conversely, suppose A satisfies the conditions (i) and (ii), then it is clearly hypothesis $A(0^g) = \tilde{X}$.

$$P_A(x+y)^g = P_A(1.x+1.y)^g \geq \min\{P_A(x^g), P_A(y^g)\}$$

$$I_A(x+y)^g = I_A(1.x+1.y)^g \leq \max\{I_A(x^g), I_A(y^g)\}$$

So, $A(x+y)^g \geq \min\{A(x^g), A(y^g)\}$ and the condition (DFNFSGM–2) of Definition-3.4 is satisfied.

Now, let us show the validity of condition (DFNFSGM–3), by the hypothesis,

$$P_A(rx)^g = P_A(rx+r0)^g \geq \min\{P_A(x^g), P_A(0^g)\} = P_A(x^g)$$

$$I_A(rx)^g = I_A(rx+r0)^g \leq \max\{I_A(x^g), I_A(0^g)\} = I_A(x^g) \text{ for each } x, y \in M, r \in R.$$

Therefore, (DFNFSGM–3) of Definition 3.4 is satisfied.

Theorem 4.12: Let A and B be double-framed N- fuzzy soft G-modules of a classical module M and G be a finite group which acts on M. Then intersection A $\cap$ B is also a DFNFSGM of M.

Proof: Since A, B $\in$ DFNFSGM(M), we have

$$A(0^g) = \tilde{X}, \quad B(0^g) = \tilde{X}.$$

$$P_{A\cap B}(0^g) = \min\{P_A(0^g), P_B(0^g)\} = 1$$

$$I_{A\cap B}(0^g) = \max\{I_A(0^g), I_B(0^g)\} = 0.$$

Let $x, y \in M$, $r, s \in R$, by Theorem 4.11, it is enough to show that

$$(A\cap B)(rx+sy)^g \geq \min\{(A\cap B)(x^g), (A\cap B)(y^g)\}$$

$$P_{A\cap B}(rx+sy)^g \geq \min\{P_{A\cap B}(x^g), P_{A\cap B}(y^g)\}$$

$$I_{A\cap B}(rx+sy)^g \leq \max\{I_{A\cap B}(x^g), I_{A\cap B}(y^g)\}$$

Now, we consider the truth-membership degree of the intersection,

$$P_{A\cap B}(rx+sy)^g = \min\{P_A(rx+sy)^g, P_B(rx+sy)^g\}$$

$$\geq \min\{\min\{P_A(x^g), P_A(y^g)\}, \min\{P_B(x^g), P_B(y^g)\}\}$$

$$= \min\{\min\{P_A(x^g), P_B(x^g)\}, \min\{P_A(y^g), P_B(y^g)\}\}$$

$$= \min\{P_{A\cap B}(x^g), P_{A\cap B}(y^g)\}$$

Then other inequalities are proved similarly.

Hence, $A \cap B \in$ DFNFSGM (M).
Note 4.13: A non-empty subset N of M is a sub module of M if and only if $rx+sy \in N$ for all $x, y \in M,\ r, s \in R$.

Proposition 4.14: Let M be a G-module over R. $A \in$ DFNFSGM(M) and G be a finite group which acts on M, if and only if for all $\alpha \in [0, 1]$, $\alpha - level\ set\ of\ A$, $(P_A)_\alpha$, $(I_A)^\alpha$ are classical G-modules of M where $A(0^g) = \tilde{X}$.
Proof: Let $A \in$ DFNFSGM(M), $\alpha \in [0, 1]$, $x, y \in (P_A)_\alpha$ *and* $r, s \in R$ *be any elements*.
Then, $P_A(x^g) \geq \alpha,\ \ P_A(y^g) \geq \alpha$ *and* $\min\{P_A(x^g),\ P_A(y^g)\} \geq \alpha$.
By using Theorem 4.11, we have,

$$P_A(rx+sy)^g \geq \min\{P_A(x)^g,\ P_A(y)^g\} \geq \alpha.$$

Hence, $rx+sy \in (P_A)_\alpha$
Therefore, $(P_A)_\alpha$ is a classical G-module of M for each $\alpha \in [0, 1]$.
Similarly, for $x, y \in (I_A)^\alpha$
We obtain $rx+sy \in (I_A)^\alpha$ for each $\alpha \in [0, 1]$.
Consequently, $(I_A)^\alpha$ is classical G-module of M for each $\alpha \in [0, 1]$.
Conversely, let $(P_A)_\alpha$ be a classical G-module of M for each $\alpha \in [0, 1]$.
Let $x, y \in M,\ \ \alpha = \min\{P_A(x)^g,\ P_A(y)^g\}$. *Then* $P_A(x)^g \geq \alpha$ *and* $P_A(y)^g \geq \alpha$.
Thus, $x, y \in (P_A)_\alpha$.
Since, $(P_A)_\alpha$ is a classical G-module of M, we have $rx+sy \in (P_A)_\alpha$ for all $r, s \in R$.
Hence, $(P_A)(rx+sy)^g \geq \alpha = \min\{P_A(x^g),\ P_A(y^g)\}$.
Now, we consider $(I_A)^\alpha$, *let* $x, y \in M,\ \ \alpha = \max\{I_A(x^g),\ I_A(y^g)\}$. *Then* $I_A(x^g) \leq \alpha,\ \ I_A(y^g) \leq \alpha$. Thus, $x, y \in (I_A)^\alpha$.
Since, $(I_A)^\alpha$ is a G-module of M, we have $rx+sy \in (I_A)^\alpha$ for all $r, s \in R$.
Thus, $(I_A)(rx+sy)^g \leq \alpha = \max\{I_A(x^g),\ I_A(y^g)\}$.
It is also obvious that $A(0^g) = \tilde{X}$. Hence the conditions of Theorem 4.11 are satisfied.

Proposition 4.15: Let A and B be two double-framed N- fuzzy soft sets X and Y respectively and G be a finite group which acts on M. Then the following equalities are satisfied for the α-level
$(P_{A\times B})_\alpha = (P_A)_\alpha \times (P_B)_\alpha$ *and* $(I_{A\times B})^\alpha = (I_A)^\alpha \times (P_B)^\alpha$.
Proof: Let $(x, y) = (P_{A\times B})_\alpha$ be arbitrary.

$$\text{So, } P_{A\times B}(x, y)^g \geq \alpha \Leftrightarrow \min\{P_A(x^g),\ P_B(y^g)\} \geq \alpha \Leftrightarrow P_A(x^g) \geq \alpha \ \ and\ \ P_B(y^g) \geq \alpha$$
$$\Leftrightarrow (x, y)^g \in (P_A)_\alpha \times (P_B)_\alpha.$$

Let $(x, y) = (I_{A\times B})^\alpha$ be arbitrary.

$$\text{Hence, } I_{A\times B}(x, y)^g \leq \alpha \Leftrightarrow \max\{I_A(x^g),\ I_B(y^g)\} \leq \alpha \Leftrightarrow I_A(x^g) \leq \alpha \ \ and\ \ I_B(y^g) \leq \alpha$$
$$\Leftrightarrow (x, y)^g \in (I_A)^\alpha \times (I_B)^\alpha.$$

Proposition 4.16: Let A, B $\in$ DFNFSGM(M). Then the product $A \times B$ is also a double-framed N-fuzzy soft G-module of M.
Proof: We know that the direct product of two soft G-modules is a G-module. So, by Proposition 4.14 and Proposition 4.15, we obtain the result.

Proposition 4.17: Let A and B be two double-framed N- fuzzy soft sets on X and Y respectively and G be a finite group which acts on M and $\phi: X \to Y$ be a mapping. Then the followings hold:

(i) $\phi\left((P_A)_\alpha\right) \subseteq \left(P_{\phi(A)}\right)_\alpha$,

$\phi\left((I_A)^\alpha\right) \supseteq \left(I_{\phi(A)}\right)^\alpha$.

(ii) $\phi^{-1}\left((P_B)_\alpha\right) = \left(P_{\phi^{-1}(B)}\right)_\alpha$,

$\phi^{-1}\left((I_B)^\alpha\right) = \left(I_{\phi^{-1}(B)}\right)^\alpha$.

Proof:(i) Let $y \in \phi\left((P_A)_\alpha\right)$, then there exists $x \in (P_A)_\alpha$ such that $\phi(x) = y$.

Hence, $P_A(x^g) \geq \alpha$. So, $\bigvee_{x\in\phi^{-1}(y)} P_A(x) \geq \alpha$, (ie) $P_{\phi(A)}(y^g) \geq \alpha$ *and* $y \in \left(P_{\phi(A)}\right)_\alpha$

Hence, $\phi\left((P_A)_\alpha\right) \subseteq \left(P_{\phi(A)}\right)_\alpha$, similarly, we obtain other inclusion.

(ii)
$$\begin{aligned}\left(P_{\phi^{-1}(B)}\right)_\alpha &= \left\{x \in X / P_{\phi^{-1}(B)}(x^g) \geq \alpha\right\}\\ &= \left\{x \in X / P_B\phi(x^g) \geq \alpha\right\}\\ &= \left\{x \in X / \phi(x) \in (P_B)_\alpha\right\}\\ &= \left\{x \in X / x \in \phi^{-1}\left((P_B)_\alpha\right)\right\}\\ &= \phi^{-1}\left((P_B)_\alpha\right).\end{aligned}$$

The other equality is obtained in a similar way.

Theorem 4.18: Let M, N be the classical G-modules and $\phi : M \to N$ be a homomorphism of G-modules. Then the pre image $\phi^{-1}(B)$ is a DFNFSGM of M.

Proof: By Proposition 4.17 (ii), we have

$\phi^{-1}\left((P_B)_\alpha\right) = \left(P_{\phi^{-1}(B)}\right)_\alpha$,

$\phi^{-1}\left((I_B)^\alpha\right) = \left(I_{\phi^{-1}(B)}\right)^\alpha$.

Since pre image of a G-module is a G-module, by Proposition 4.14, we obtain the result.

Corollary 4.19: If $\phi : M \to N$ is a homomorphism of G-modules and $\{B_j : j \in I\}$ is a family of double-framed N- fuzzy soft G-modules of N, then the image $\phi^{-1}\left(\bigcap B_j\right)$ is a DFNFSGM of M.

Theorem 4.20: Let M and N be the classical G-modules and $\phi : M \to N$ be a homomorphism of G-modules. If 'A' is a DFNFSGM of M and G is a finite group which acts on M, then the image $\phi(A)$ is a DFNFSGM of N.

Proof: By Proposition 4.14, it is enough to show that $\left(P_{\phi(A)}\right)_\alpha, \left(I_{\phi(A)}\right)^\alpha$ are G-sub modules of N for all $\alpha \in [0, 1]$.

Let $y_1, y_2 \in \phi\left((P_A)_\alpha\right)$.Then $P_{\phi(A)}({y_1}^g) \geq \alpha$ *and* $P_{\phi(A)}({y_2}^g) \geq \alpha$, there exist $x_1, x_2 \in M$ such that $P_A({x_1}^g) \geq \alpha,\ P_{\phi(A)}({y_1}^g) \geq \alpha$ *and* $P_A({x_2}^g) \geq \alpha,\ P_{\phi(A)}({y_2}^g) \geq \alpha.$

Then $P_A({x_1}^g) \geq \alpha,\ \ P_A({x_2}^g) \geq \alpha$ *and* $\min\left\{P_A({x_1}^g), P_A({x_2}^g)\right\} \geq \alpha.$

Since 'A' is a DFNFSGM of M, for any $r, s \in R$, we have

$P_A(rx_1 + sx_2)^g \geq \min\left\{P_A(x_1)^g, P_A(x_2)^g\right\} \geq \alpha$

Hence, $rx_1 + sx_2 \in (P_A)_\alpha \Rightarrow \phi(rx_1 + sx_2) \in \phi(P_A)_\alpha \subseteq \left(P_{\phi(A)}\right)_\alpha \Rightarrow r\,\phi(x_1) + s\,\phi(x_2) \in \left(P_{\phi(A)}\right)_\alpha$

$\Rightarrow ry_1 + sy_2 \in \left(P_{\phi(A)}\right)_\alpha$.

Therefore, $\left(P_{\phi(A)}\right)_\alpha$ is a G-sub module of N.

Similarly, $\left(I_{\phi(A)}\right)^{\alpha}$ is also a classical G-sub module of N for each $\alpha \in [0, 1]$.

By Proposition 4.14, $\phi(A)$ is a DFNFSGM of N.

Corollary 4.21: If $\phi : M \to N$ is a surjective G-module homomorphism and $\{A_i ; i \in I\}$ is a family of double-framed N- fuzzy soft G-modules of M, then the image $\phi^{-1}\left(\bigcap A_i\right)$ is a DFNFSGM of N.

***Definition 4.22*:** Let A be a DFNFSGM of M and G be a finite group which acts on M. Then 'A' is said to be G-invariant DFNFSGM of M if and only if $A^g(x) = A(x^g) \geq A(x),\ for\ all\ x \in M,\ g \in G.$

Theorem 4.24: Let M and M' be Z-modules which G acts on M and let 'f' be a bijective G-module homomorphism from M. Then $f(A)$ is a G-invariant DFNFSGM of M.

Proof: Since 'A' is a G-invariant DFNFSGM of M', therefore $A^g = A,\ for\ g \in G.$

Now, $(f(A))^g = f(A^g) = f(A), \forall g \in G.$

Hence, $f(A)$ is G-invariant DFNFSGM of M'.

CONCLUSION

Modules over a ring are a generalisation of abelian groups (which are modules over Z) [Hungerford.T.W]. From the philosophical point of set view, it has been shown that a double-framed N- fuzzy soft set generalizes a classical set, fuzzy set, interval valued fuzzy set, intuitionistic fuzzy set ,etc. A double-framed N- fuzzy soft set is an instance of double-framed N- fuzzy set which can be used in real scientific and E-generating problems.

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