Linear Connections On Manifold Admitting F(2k + 5, 5) - Structure

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Abstract. D. Demetropoulou [2] and others have studied linear connections in the manifold admitting $f(2\nu+3,-1)$-structure. The aim of the present paper is to study some properties of linear connections in a manifold admitting $F(2K+5,5)$-structure. Certain interesting results have been obtained.

1. Preliminaries

Let F be a non-zero tensor field of the type $(1,1)$ and of class C^∞ on an n-dimensional manifold M^n such that [5, 8]

$$F^{2K+5} + F^5 = 0, \tag{1.1}$$

where K is a fixed positive integer greater than or equal to 1. The rank of $(F) = r$ =constant.

Let us define the operators on M as follows [5, 8]

$$l = -F^{2K}, \qquad m = I + F^{2K} \tag{1.2}$$

where I denotes the identity operator.
We will state the following two theorems [5]

Theorem 1.1. *Let M^n be an F-structure manifold satisfying (1.1), then*

$$\begin{cases} a. & l + m = I, \\ b. & l^2 = l, \\ c. & m^2 = m, \\ d. & lm = ml = 0. \end{cases} \tag{1.3}$$

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Thus for $(1,1)$ tensor field $F(\neq 0)$ satisfying (1.1), there exist complementary distributions D_l and D_m corresponding to the projection operators l and m respectively. Then, dim $D_l = r$ and dim $D_m = (n-r)$.

Theorem 1.2. *We have*

$$\begin{cases} a. & lF = Fl = F, \qquad mF = Fm = 0. \\ b. & F^{2K}m = 0, \qquad F^{2K}l = -l. \end{cases} \tag{1.4}$$

Thus F^K acts on D_l as an almost complex structure and on D_m as a null operator.
Let us define the operators $\overline{\nabla}$ and $\widetilde{\nabla}$ on M^n in terms of an arbitrary connections ∇ as under

$$\overline{\nabla}_X Y = l\nabla_X(mY) + m\nabla_X(lY) \tag{1.5}$$

and

$$\widetilde{\nabla}_X Y = l\nabla_{lX}(mY) + m\nabla_{mX}(lY) + l[lX, mY] + m[mX, lY] \tag{1.6}$$

Then it is easy to show that $\overline{\nabla}$ and $\widetilde{\nabla}$ are linear connections on the manifold M^n [2]

2. Distributions anti-parallelism and anti-half parallelism

In this section, first we have the following definitions:

Definition 2.1. *Let us call the distribution D_L as ∇-anti parallel if for all TM^n denotes the tangent bundle of the manifold M^n.*

Definition 2.2. *The distribution D_L will be called ∇ anti-half parallel if for all $X \in D_L$ and $Y \in TM^n$, the vector field $\nabla_Y X \in D_M$, where*

$$(\triangle F)(X,Y) = F\nabla_X Y - F\nabla_Y X - \nabla_{FX} Y + \nabla_Y FX \tag{2.1}$$

F being a $(1,1)$ tensor field on M^n satisfying the equation (1.1).

In a similar manner, anti-half parallelism of the distribution D_M can also be defined.

Theorem 2.3. *In the $F(2K+5,5)$-structure manifold M^n, the distribution D_L and D_M are anti-parallel with respect to connections $\overline{\nabla}$ and $\widetilde{\nabla}$.*

Proof. Let $X \in TM^n$ and $Y \in D_L$, therefore $mY = 0$. Hence in view of equation (1.5), we get

$$\overline{\nabla}_X Y = m\nabla_X(lY) \in D_M.$$

Hence the distribution D_L is anti-half parallel with respect to the linear connection $\overline{\nabla}$. Similarly, it can also be shown that D_M is also $\overline{\nabla}$ is also anti-parallel.
Again in view of the equation (1.5), taking $mY = 0$, we obtain

$$\widetilde{\nabla}_X Y = m\nabla_{mX}(lY) + m[mX, lY] \in D_M. \tag{2.2}$$

Thus the distribution D_L is anti-parallel with respect to the linear connection $\widetilde{\nabla}$. A similar result for D_M can also be proved in a similar manner. ■

Theorem 2.4. *In the $F(2K+5,5)$-structure manifold M^n, the distribution D_L and D_M are anti-parallel with respect to connection ∇ if and only if ∇ and $\overline{\nabla}$ are equal.*

Proof. Since the distributions D_L and D_M are anti-parallel with respect to the ∇, hence

$$l\nabla_X(lY) = m\nabla_X(mY) = 0, \tag{2.3}$$

for the vector fields $X,\ Y\ \in TM^n$.
Since $l + m = I$, hence in view of equation (2.3), it follows that

$$\begin{aligned} \nabla_X(lY) &= m\nabla_X(lY), \\ \nabla_X(mY) &= l\nabla_X(mY) \end{aligned} \tag{2.4}$$

Thus in view of the equations (1.5) and (2.4), it follows that

$$\overline{\nabla}_X Y = \nabla_X Y.$$

Hence, the connections ∇ and $\overline{\nabla}$ are equal.
The converse can also be proved easily. ■

Theorem 2.5. *In a $F(2K+5,5)$-structure manifold M^n, the distribution D_M is anti-half parallel with respect to connection $\overline{\nabla}$ if*

$$m\nabla_{FX}(lY) = m\overline{\nabla}_Y(FX), \tag{2.5}$$

for arbitrary $X \in D_M$ and $Y \in TM^n$.

Proof. Since $mF = Fm = 0$, hence in view of the equation (2.1), we get for the connection $\overline{\nabla}$

$$m(\triangle F)(X,Y) = m\overline{\nabla}_Y(FX) - m\nabla_{FX}(Y). \tag{2.6}$$

By virtue of the equation (1.5), the above equation (2.6) takes the form

$$\begin{aligned} m(\triangle F)(X,Y) &= m\{l\overline{\nabla}_Y(mFX) + m\overline{\nabla}_Y(lFX)\} \\ &- m\{l\nabla_{FX}(mY) + m\nabla_{FX}(lY)\}. \end{aligned} \tag{2.7}$$

Since, $ml = lm = 0$; $Fl = lF = F$ and m is the projection operator, the above equation (2.7) takes the form,

$$m(\triangle F)(X, Y) = m\overline{\nabla}_Y(FX) - m\nabla_{FX}(lY). \qquad (2.8)$$

Since the distribution D_M is $\overline{\nabla}$ anti-half parallel so far all $X \in D_M$, $Y \in TM^n$,

$$m(\triangle F)(X, Y) \in D_L.$$

Thus,

$$m\overline{\nabla}_Y(FX) = m\nabla_{FX}(lY).$$

Hence, the theorem is proved. ∎

Theorem 2.6. *In the manifold M^n equipped with $F(2K+5, 5)$-structure, the distribution D_L is anti-half parallel with respect to the connection $\overline{\nabla}$ if*

$$F\nabla_X(lY) = l\nabla_{FX}(mY),$$

for arbitrary $X \in D_L$ and $Y \in TM^n$.

Proof. Proof follows easily in a way similar to that of the theorem **2.5**. ∎

Theorem 2.7. *In the $F(2K+5, 5)$-structure manifold M^n, the distribution D_M is anti-half parallel with respect to the connection $\widetilde{\nabla}$ if for $X \in D_M$ and $Y \in TM^n$ the equation*

$$m\nabla_{mY}(FX) + m[mY, FX] = 0$$

is satisfied.

Proof. For $X \in D_M$ and $Y \in TM^n$, we have for the connection $\widetilde{\nabla}$

$$(\triangle F)(X, Y) = F\widetilde{\nabla}_X Y - F\widetilde{\nabla}_Y X - \widetilde{\nabla}_{FX} Y + \widetilde{\nabla}_Y FX. \qquad (2.9)$$

As $Fm = mF = 0$, hence from the above equation (2.9), it follows that

$$m(\triangle F)(X, Y) = m\widetilde{\nabla}_Y FX - m\widetilde{\nabla}_{FX} Y. \qquad (2.10)$$

In view of the equation (1.4) and (1.6), it is easy to show that

$$m\widetilde{\nabla}_{FX} Y = 0 \qquad (2.11)$$

and

$$m\widetilde{\nabla}_Y FX = m\nabla_{mY}(FX) + m[mY, FX]. \qquad (2.12)$$

Thus, we get

$$m(\triangle F)(X, Y) = m\nabla_{mY}(FX) + m[mY, FX]. \qquad (2.13)$$

The distribution D_M will be $\tilde{\tilde{\nabla}}$ anti-half parallel if $X \in D_M$, $Y \in TM^n$, the vector field $(\triangle F)(X,Y) \in D_L$. Thus,

$$m(\triangle F)(X,Y) = 0$$

i.e.,

$$m\nabla_{mY}(FX) + m[mY, FX] = 0.$$

Hence, the theorem is proved. ■

3. Geodesic in the manifold M^n

Let C be a curve in M^n, T a tangent field and ∇ arbitrary connection on M^n. Then, we have

Definition 3.1. *The curve C is a geodesic with respect to the connection ∇ if $\nabla_T T = 0$ along C.*

Applying the definition for the connection $\overline{\nabla}$ and $\tilde{\tilde{\nabla}}$, we have the following results in the $F(2K+5,5)$-structure manifold M^n.

Theorem 3.2. *A curve C is a geodesic in the manifold M^n with respect to the connection ∇ if the vector fields*

$$\nabla_T T - \nabla_T(lT) \in D_M \quad \textit{and} \quad \nabla_T(lT) \in D_L.$$

Proof. The curve C will be $\overline{\nabla}$ geodesic if $\overline{\nabla}_T T = 0$.
In view of the equation (1.5), the above equation takes the form

$$l\nabla_T(I-l)T + m\nabla_T(lT) = 0$$

or equivalently

$$l\nabla_T T - l\nabla_T(lT) + m\nabla_T(lT) = 0,$$

which implies that

$$\nabla_T T - \nabla_T(lT) \in D_M \quad \text{and} \quad \nabla_T(lT) \in D_L.$$

This proves the theorem. ■

Theorem 3.3. *A curve C is a geodesic in the manifold M^n with respect to the connection ∇ if*

$$\nabla_{lT} T - \nabla_{lT}(lT) + [lT, mT] \in D_M \quad \textit{and} \quad \nabla_{mT}(lT) + [mT, lT] \in D_L.$$

Proof. Using definition of ∇ from the equation (1.6), proof follows easily as of theorem **3.2**. ■

Theorem 3.4. *The $(1,1)$ tensor field l is covariant constant with respect to the connection $\overline{\nabla}$ if*

$$m\nabla_X(lY) = l\nabla_X(mY) \tag{3.1}$$

but the tensor field m is always covariant constant.

Proof. We have

$$(\overline{\nabla}_X l)Y = \overline{\nabla}_X(lY) - l\overline{\nabla}_X Y \tag{3.2}$$

In view of equation (1.5), the above equation takes the form

$$\begin{aligned}(\overline{\nabla}_X l)Y &= l\nabla_X(mlY) + m\nabla_X(lY) \\ &- l\{l\nabla_X(mY) + m\nabla_X(lY)\}.\end{aligned} \tag{3.3}$$

Since $l^2 = l$ and $lm = ml = 0$, the equation (3.3) takes the form

$$(\overline{\nabla}_X l)Y = m\nabla_X(lY) - l\nabla_X(mY). \tag{3.4}$$

The $(1,1)$ tensor field l is covariant with respect to the connection $\overline{\nabla}$ if

$$(\overline{\nabla}_X l)Y = 0. \tag{3.5}$$

Hence in view of the equation (3.4) and (3.5), we get

$$m\nabla_X(lY) = l\nabla_X(mY).$$

This proves the first part of the theorem.

Again it can be easily shown that

$$(\overline{\nabla}_X m)Y = 0,$$

for all vector fields $X,\ Y \in TM^n$. Thus, the tensor field m is always covariant constant. ■

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