Stability Analysis of Three Dimensional Advection-Diffusion Equation with a Mixed Derivative

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Abstract

The paper studies stability analysis for (3+1) Dimensional Advection-Diffusion equation with a mixed derivative using the method of Von-Neuman. Taylor series expansion has been used to generate the Alternating Direction Explicit scheme (ADE) and Alternating Direction Implicit scheme (ADI) and the schemes have been found to be unconditionally stable.

Keywords: *(3+1) Dimensional Advection Diffusion Equation, Partial Differential Equations (PDE'S), Alternating Direction Explicit (ADE) scheme, Alternating Direction Implicit (ADI) scheme,Amplification factor.*

Mathematics Subject Classification: Primary 65N30, 65M12, 65M06; Secondary 65D05, 65M22, 65M60

Introduction

Use of Advection – Diffusion equation in various fields of science like transport of heat, sediment, ground water and surface flow pollutants are fully sufficient for researchers to show interest in solving this equation. Many researchers like Elder [2] tried to propose analytical solutions for these type of equations, but in recent years researchers like Gunn [10] have shown more interest thereby introducing numerical solutions to these kind of equations. As noted earlier, most of the researchers showed interest to present numerical solutions for Advection – Diffusion Equation instead of analytical solutions.The combination of hydrodynamic dispersion and molecular diffusion coefficients is considered to describe the solute transport – Essa [4]. The dominant process of solute transport is advection moving aqueous chemical species along with fluid flow. Most of the solute transport modelling begins with advection transport. The advection – dispersion equation describes the spatial and temporal variation in solute concentration with specific initial and boundary conditions. The governing equation known as the constant parameter advection – dispersion equation may be derived for the case of steady and unsteady flows.The traditional advection – dispersion equation represents a standard model to predict the solute concentration in an aquifer which is based on conservation of mass and Fick's law of diffusion. Gunn [10] explored the dispersion theory based on a relationship between two parameters namely dispersion co-efficient and seepage velocity and proposed two possible relationships:

(i) The dispersion coefficient is proportional to seepage velocity
(ii) The dispersion co efficient is proportional to the square of seepage velocity.

A general theory of dispersion in porous media was explored by Ebach [1]. Later, the dispersion theory was generalized as dispersion coefficients proportional to the power of seepage velocity where power ranges from 1 to 2 as demonstrated by Griths [9]. Ghosh [7] also experimentally observed that the dispersion coefficient is directly proportional to the seepage velocity with a power ranging from 1 to 1.2. Gottlieb [8] described the time-dependent input concentration for longitudinal dispersion flow.

Brief review of work done by attention to the data was done by Ebach [1] who developed an algorithm to solve fully conservative, high resolution Advection – Diffusion Equation in irregular geometries. In this algorithm they developed Finite Volume Method to solve this equation. Elder [2] in order to numerically integrate the semi – discrete equation arising arising after the spatial discretization of Advection – Reaction – Diffusion Equation applied two variable step linearly implicit Runge-Kutta methods of order 3 and 4 equations.Parabolic partial differential equations in three space dimensions with over-specified boundary data feature in the mathematical modeling of many importantphenomena. While a significant body of knowledge about the theory and numerical methods for parabolic partial differential equations with classical boundary conditions has been accumulated, not much has been extended to parabolic partial differential equations with over-specified boundary data [4]. We often meet the problem of solving equation of parabolic type in many fields such as seepage, diffusion, heat conductionand so on[9].Lapidus, L.[12] and Sharma,K.D [7] used ADI to solve the two dimensional time dependentheat equations subject to a constant coefficient Fischer H.B [5]used ADI methods for solving elliptic problems.

Johnson, S.[11] used the Euclerian-Lagrangian localized adjoin method on non – uniform time steps and unstructured meshes to solve the Advection – Diffusion Equation. Griths [9] tried to develop an algorithm by second and third order accuracy with finite with finite – difference method to solve the convection – diffusion equation. In this algorithm they used the counter error mechanism to reduce numerical dispersion. One of the researchers that tried to solve Advection – Diffusion Equation in implicit condition is Gottlieb [8]. He solved the equation with Finite Difference Method by using the upwind and Crank – Nicolson schemes.In this paperwe derive the finite difference forms of ADE and ADI methods for the given model equation and then analyse the algorithm for each method.

The model equation

The research examines the consistency of the Alternating Direction Explicit (ADE) scheme and Alternating Direction Implicit (ADI) scheme for solving the (3+1) Dimensional Advection-Diffusion.

$$f_1(x,y,z,t)\frac{\partial^2 C}{\partial x^2}+f_2(x,y,z,t)\frac{\partial^2 C}{\partial y^2}+f_3(x,y,z,t)\frac{\partial^2 C}{\partial z^2}+f_4(x,y,z,t)\frac{\partial^2 C}{\partial x\partial y}+f_5(x,y,z,t)+f_6(x,y,z,t)\frac{\partial C}{\partial y}=C_t \quad (1)$$

Which is used to model physical process of advection diffusion in a (3+1) Dimensional system such as one involving contaminant concentration in aquifer. The coefficients $f_1(x,y,z,t), f_2(x,y,z,t), f_3(x,y,z,t)$ represent the diffusion parameters (diffusivity) and $f_4(x,y,z,t) and f_5(x,y,z,t)$ are the advection parameters (velocity). The equation is parabolic and is derived from the principle of conservation of mass using Fick's law of conservation in fluid flow problems as presented by (Morton 1971).The Alternating Direction Explicit (ADE) scheme developed for the equation is given by:-

$$4qC_{i,j,k}^{n+1}=4C_{i+1,j,k}^{n}-24C_{i,j,k}^{n}+4C_{i-1,j,k}^{n}+4C_{i,j+1,k}^{n}+4C_{i,j-1,k}^{n}+4C_{i,j,k+1}^{n}4C_{i,j,k-1}^{n}+C_{i+1,j+1,k}^{n}-C_{i+1,j-1,k}^{n}-C_{i-1,j+1,k}^{n}+C_{i-1,j-1,k}^{n}+2qC_{i+1,j,k}^{n}+2qC_{i,j+1,k}^{n} \quad (2)$$

and the Alternating Direction Implicit (ADI) scheme developed for the equation is given by:-

$$4qC_{i,j,k}^{n+1}+4C_{i-1,j,k}^{n+1}-8C_{i,j,k}^{n+1}-4C_{i+1,j,k}^{n+1}=4qC_{i,j,k}^{n}-16C_{i,j,k}^{n}+4C_{i,j+1,k}^{n}+4C_{i,j-1,k}^{n}+4C_{i,j,k+1}^{n}+4C_{i,j,k-1}^{n}+C_{i+1,j+1,k}^{n}-C_{i+1,j-1,k}^{n}-C_{i-1,j+1,k}^{n}+C_{i-1,j-1,k}^{n}+2qC_{i+1,j,k}^{n}-2qC_{i-1,j,k}^{n}+2qC_{i,j+1,k}^{n}-2qC_{i,j-1,k}^{n} \quad (3)$$

Properties of numerical schemes

Many techniques are available for numerical simulation work and in order to quantify how well a particular numerical technique performs in generating a solution to a problem, there are four fundamental criteria that can be applied to compare and contrast different methods. The concepts are accuracy, consistency, stability and convergence. The method of Finite Difference Method is one of the most valuable methods of approximating numerical solution of Partial Differential Equations (PDEs). Before numerical computations are made, these four important properties of finite difference equations must be considered.

(a) **Accuracy**: Is a measure of how well the discrete solution represents the exact solution of the problem. Two quantities exist to measure this, the local or truncation error, which measures how well the difference equations match the differential equations, and the global error which reacts to the overall error in the solution. This is not possible to find unless the exact solution is known.

(b) **Stability**: A finite difference scheme is stable if the error made at one time step of the calculation do not cause the errors to be magnified as the computations are continued. A neutrally stable scheme is one in which errors remain constant as the computation are carried forward. If the errors decay are eventually damp out, the numerical scheme is said to be stable. If on the contrary, the errors grow with time the numerical scheme is said to be unstable.

(c) **Consistency**: When a truncation error goes to zero, a finite difference equation is said to be consistent or compatible with a partial differential equation. Consistency requires that the original equations can be recovered from the algebraic equations. Obviously this is a minimum requirement for any discretization.

(d) **Convergence**: A solution of a set of algebraic equations is convergent if the approximate solution approaches the exact solution of the Partial Differential Equations (PDEs) for each value of the independent variable. For example, as the mesh sizes approaches zero, the grid spacing and time step also goes to zero.

Lax had proved that under appropriate conditions a consistent scheme is convergent if and only if it is stable. According to $Lax - RichtmyerEquivalenceTheorem$ which states that "given a properly posed linear initial value problem and a finite difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence"

Stability of numerical schemes

Stability considerations are very important in getting the numerical solution of a differential equation using finite difference methods. The solution of the finite difference equation is said to be stable, if the error do not growexponentially as we progress from one step to another. We analyse stability for two numerical schemes developed for equation (1) namely Alternating Direction Explicit (ADE) and Alternating Direction Implicit (ADI) numerical schemes using the method attributed to John Von Neuman and Jean Baptiste Joseph Fourier. The primary observation in the Fourier(Von Neuman) method is that the Numerical scheme is linear and therefore it will have a solution in the form $C(x,t) = \lambda^t e^{i\propto x}$. Thus the numerical scheme is stable provided $|\lambda| \leq 1$ and unstable provides $|\lambda| \geq 1$ where λ is referred to as the Amplification factor.

Stability analysis of the ADE scheme

Von Neumanmethod is used to investigate stability of parabolic methods in three dimensions. We shall apply this method by substituting the solution in Finite Difference method at the time t by $C_{i,j,k}^n = \psi(t)e^{m\propto x}e^{m\beta y}e^{m\gamma x}$ where $\propto, \beta$ and $\gamma \geq 0$ and $m = \sqrt{-1}$, [4,10]. To apply Von Neuman to the ADE scheme, we will have to linearlize the scheme as explained by Gunn[27]

$$4qC_{i,j,k}^{n+1} = 4C_{i+1,j,k}^n + 4C_{i-1,j,k}^n - 24C_{i,j,k}^n + 4C_{i,j+1,k}^n + 4C_{i,j-1,k}^n + 4C_{i,j,k+1}^n + 4C_{i,j,k-1}^n + C_{i+1,j+1}^n - C_{i+1,j-1}^n - C_{i-1,j+1}^n + C_{i-1,j-1}^n + 2qC_{i+1,j,k}^n + 2qC_{i,j+1,k}^n \quad (4)$$

Where $q = r = \frac{\Delta t^2}{\Delta x}$ and we assume that $C_{ijk}^n = \psi(t)e^{m\propto x}e^{m\beta y}e^{m\gamma z}$. Substituting in equation (4) will yield

$$4q\psi(t+\Delta t)e^{m\propto x}e^{m\beta y}e^{m\gamma z} = 4\psi(t)e^{m\propto(x+\Delta x)}e^{m\beta y}e^{m\gamma z} - 24\psi(t)e^{m\propto(x+\Delta x)}e^{m\beta y}e^{m\gamma z} + 4\psi(t)e^{m\propto(x-\Delta x)}e^{m\beta y}e^{m\gamma z} + 4\psi(t)e^{m\propto x}e^{m\beta(y+\Delta y)}e^{m\gamma z} + 4\psi(t)e^{m\propto x}e^{m\beta(y-\Delta y)}e^{m\gamma z} + 4\psi(t)e^{m\propto x}e^{m\beta(y-\Delta y)}e^{m\gamma z} + 4\psi(t)e^{m\propto x}e^{m\beta y}e^{m\gamma(z+\Delta z)} + 4\psi(t)e^{m\propto x}e^{m\beta y}e^{m\gamma(z-\Delta z)} + \psi(t)e^{m\propto(x+\Delta x)}e^{m\beta(y+\Delta y)}e^{m\gamma z} - \psi(t)e^{m\propto(x+\Delta x)}e^{m\beta(y-\Delta y)}e^{m\gamma z} + \psi(t)e^{m\propto(x-\Delta x)}e^{m\beta(y+\Delta y)}e^{m\gamma z} + \psi(t)e^{m\propto(x-\Delta x)}e^{m\beta(y-\Delta y)}e^{m\gamma z} + 2q\psi(t)e^{m\propto(x+\Delta x)}e^{m\beta y}e^{m\gamma z} + 2q\psi(t)e^{m\propto x}e^{m\beta(y+\Delta y)}e^{m\gamma z} .(5)$$

Dividing equation (5) by $\psi(t)e^{m\propto x}e^{m\beta y}e^{m\gamma z}$ will reduce it to

$$\frac{4q\psi(t+\Delta t)}{\psi t} = \lambda = 4e^{m\propto\Delta x} - 24 + 4e^{-m\propto\Delta x} + 4e^{m\beta\Delta y} + 4e^{-m\beta\Delta y} + 4e^{m\gamma\Delta z} + 4e^{-m\gamma\Delta z} + e^{m\propto\Delta x}e^{m\beta\Delta y} - e^{m\propto\Delta x}e^{-m\beta\Delta y} - e^{-m\propto\Delta x}e^{m\beta\Delta y} + e^{-m\propto\Delta x}e^{-m\beta\Delta y} + 2qe^{m\propto\Delta x} + 2qe^{m\beta y\Delta y} \quad (6)$$

By using Euler's formulae,

(i) $e^{m\propto\Delta x} = Cos(\propto\Delta x) + mSin(\propto\Delta x)$
(ii) $e^{-m\propto\Delta x} = Cos(\propto\Delta x) - mSin(\propto\Delta x)$
(iii) $e^{m\beta\Delta y} = Cos(\beta\Delta y) + mSin(\beta\Delta y)$
(iv) $e^{-m\beta\Delta y} = Cos(\beta\Delta y) - mSin(\beta\Delta y)$
(v) $e^{m\gamma\Delta z} = Cos(\gamma\Delta z) + mSin(\gamma\Delta z)$
(vi) $e^{-m\gamma\Delta z} = Cos(\gamma\Delta z) - mSin(\gamma\Delta z)$

And also by applying trigonometry double angle formulae

(i) $Sin(\propto \Delta x) = 2\, Sin\left(\frac{\propto \Delta x}{2}\right) Cos\left(\frac{\propto \Delta x}{2}\right)$

(ii) $Cos(\propto \Delta x) = 1 - 2\, Sin^2\left(\frac{\propto \Delta x}{2}\right)$

Substituting in the equation (6) will now yield

$$\frac{4q\psi(t+\Delta t)}{\psi t} = 8\,Cos(\propto \Delta x) + 8\,Cos(\beta \Delta y) + 8\,Cos(\gamma \Delta z) + 4mSin\left(\frac{\beta \Delta y}{2}\right) Cos\left(\frac{\beta \Delta y}{2}\right) - 4mSin\left(\frac{\beta \Delta y}{2}\right) Cos\left(\frac{\beta \Delta y}{2}\right) Sin^2\left(\frac{\alpha \Delta x}{2}\right) - 8\,Sin\left(\frac{\alpha \Delta x}{2}\right) Cos\left(\frac{\alpha \Delta x}{2}\right) Sin\left(\frac{\beta \Delta y}{2}\right) Cos\left(\frac{\beta \Delta y}{2}\right) - 2 + 2Sin^2\left(\frac{\alpha \Delta x}{2}\right) - 2Sin^2\left(\frac{\beta \Delta y}{2}\right) + Sin^2\left(\frac{\beta \Delta y}{2}\right) Sin^2\left(\frac{\alpha \Delta x}{2}\right) - 2mSin\left(\frac{\alpha \Delta x}{2}\right) Cos\left(\frac{\alpha \Delta x}{2}\right) + 4mSin\left(\frac{\alpha \Delta x}{2}\right) Cos\left(\frac{\alpha \Delta x}{2}\right) Cos\left(\frac{\beta \Delta y}{2}\right) + 2q - 4qSin^2\left(\frac{\alpha \Delta x}{2}\right) + 4qmSin\left(\frac{\alpha \Delta x}{2}\right) Cos\left(\frac{\alpha \Delta x}{2}\right) + 2q - 4qSin^2\left(\frac{\beta \Delta y}{2}\right) + 4qmSin\left(\frac{\beta \Delta y}{2}\right) - \left(\frac{\beta \Delta y}{2}\right) - 24 \quad (7)$$

Equation (7) further reduces to

$$\lambda = \frac{\psi(t+\Delta t)}{\psi t} = \frac{\dfrac{8-16Sin^2\left(\frac{\propto \Delta x}{2}\right)+8-16Sin^2\left(\frac{\beta \Delta y}{2}\right)+8-16Sin^2\left(\frac{\gamma \Delta z}{2}\right)+4mSin\left(\frac{\beta \Delta y}{2}\right)Cos\frac{(\beta \Delta y)}{2}-}{4mSin\left(\frac{\beta \Delta y}{2}\right)Cos\left(\frac{\beta \Delta y}{2}\right)Sin^2\left(\frac{\propto \Delta x}{2}\right)-8\,Sin\left(\frac{\propto \Delta x}{2}\right)Cos\left(\frac{\propto \Delta x}{2}\right)Sin\left(\frac{\beta \Delta y}{2}\right)Cos\left(\frac{\beta \Delta y}{2}\right)-2+}}{\dfrac{2Sin^2\left(\frac{\propto \Delta x}{2}\right)-2Sin^2\left(\frac{\beta \Delta y}{2}\right)+Sin^2\left(\frac{\beta \Delta y}{2}\right)Sin^2\left(\frac{\propto \Delta x}{2}\right)-2mSin\left(\frac{\propto \Delta x}{2}\right)Cos\left(\frac{\propto \Delta x}{2}\right)+}{\frac{4mSin\left(\frac{\propto \Delta x}{2}\right)Cos\left(\frac{\propto \Delta x}{2}\right)Cos^2\left(\frac{\beta \Delta y}{2}\right)+2q-4q\,Sin^2\left(\frac{\propto \Delta x}{2}\right)+4qm\,Sin\left(\frac{\propto \Delta x}{2}\right)Cos\left(\frac{\propto \Delta x}{2}\right)+2q-4qSin^2\left(\frac{\beta \Delta y}{2}\right)+4qmSin\left(\frac{\beta \Delta y}{2}\right)Cos\left(\frac{\beta \Delta y}{2}\right)-24}{4q}}}$$

(8)

And for stability requirement

$$\left|\frac{\psi(t+\Delta t)}{\psi t}\right| = \left|\frac{\dfrac{8-16Sin^2\left(\frac{\propto \Delta x}{2}\right)+8-16Sin^2\left(\frac{\beta \Delta y}{2}\right)+8-16Sin^2\left(\frac{\gamma \Delta z}{2}\right)+4mSin\left(\frac{\beta \Delta y}{2}\right)Cos\frac{(\beta \Delta y)}{2}-}{4mSin\left(\frac{\beta \Delta y}{2}\right)Cos\left(\frac{\beta \Delta y}{2}\right)Sin^2\left(\frac{\propto \Delta x}{2}\right)-8\,Sin\left(\frac{\propto \Delta x}{2}\right)Cos\left(\frac{\propto \Delta x}{2}\right)Sin\left(\frac{\beta \Delta y}{2}\right)Cos\left(\frac{\beta \Delta y}{2}\right)-2+}}{\dfrac{2Sin^2\left(\frac{\propto \Delta x}{2}\right)-2Sin^2\left(\frac{\beta \Delta y}{2}\right)+Sin^2\left(\frac{\beta \Delta y}{2}\right)Sin^2\left(\frac{\propto \Delta x}{2}\right)-2mSin\left(\frac{\propto \Delta x}{2}\right)Cos\left(\frac{\propto \Delta x}{2}\right)+}{\frac{4mSin\left(\frac{\propto \Delta x}{2}\right)Cos\left(\frac{\propto \Delta x}{2}\right)Cos^2\left(\frac{\beta \Delta y}{2}\right)+2q-4q\,Sin^2\left(\frac{\propto \Delta x}{2}\right)+4qm\,Sin\left(\frac{\propto \Delta x}{2}\right)Cos\left(\frac{\propto \Delta x}{2}\right)+2q-4qSin^2\left(\frac{\beta \Delta y}{2}\right)+4qmSin\left(\frac{\beta \Delta y}{2}\right)Cos\left(\frac{\beta \Delta y}{2}\right)-24}{4q}}}\right| \leq 1 \quad (9)$$

We analyse the algorithm of (9). This is done for the maximum and minimum Amplification Factor (λ)

(a) For maximum value of λ we take $\left(\frac{\propto \Delta x}{2}\right), \left(\frac{\beta \Delta x}{2}\right), \left(\frac{\gamma \Delta x}{2}\right)$ to be equal to zero, hence

$$|\lambda| = \left|\frac{\psi(t+\Delta t)}{\psi(t)}\right| = \left|\frac{8+8+8-2+2q+2q-24}{4q}\right| \leq 1 \qquad (10)$$

Equation (10) further reduces to

$|\lambda| = \left|\frac{4q-2}{4q}\right| \leq 1$, (Denominator is greater than the Numerator).(11)

(b) For minimum value of λ we take $\left(\frac{\propto \Delta x}{2}\right), \left(\frac{\beta \Delta x}{2}\right), \left(\frac{\gamma \Delta x}{2}\right)$ to be equal to 90^0, hence

$$|\lambda| = \left|\frac{\psi(t+\Delta t)}{\psi(t)}\right| = \left|\frac{8-16+8-16+8-16-2+2-2+1-4q-24}{4q}\right| \leq 1 \quad (12)$$

Equation (12) in this case reduces to $|\lambda| = \left|\frac{-(49+4q)}{4q}\right| \leq 1$,(Denominator is greater than theNumerator). Therefore the explicit scheme is stable. Obviously $|\lambda|$ will always be less than *one* in equations (11) and (12). All these conditions are satisfied as the left hand side of the inequalities required. Thus the ADE scheme is stable for all the values of q ie Unconditionally Stable.

Stability Analysis of the ADI Scheme

The ADIscheme generated can be written as

$$4C_{i+1,j,k}^{n+1} + 4C_{i-1,j,k}^{n+1} - 8C_{i,j,k}^{n+1} + 4C_{i,j+1,k}^{n} + 4C_{i,j-1,k}^{n} + 4C_{i,j,k+1}^{n} + 4C_{i,j,k-1}^{n} + C_{i+1,j+1,k}^{n} - C_{i+1,j-1,k}^{n} + C_{i-1,j-1,k}^{n} - C_{i-1,j+1,k}^{n} + 2qC_{i+1,j,k}^{n} - 2q(C_{i-1,j,k}^{n}) + 2qC_{i,j+1,k}^{n} - 2qC_{i,j-1,k}^{n} - 4qC_{i,j,k}^{n+1} - 4qC_{i,j,k}^{n} - 16qC_{i,j,k}^{n} = 0 \quad (13)$$

Where $q = \frac{\Delta t}{(\Delta x)^2}$ and we assume that$C_{ijk}^{n} = \psi(t)e^{m\propto x}e^{m\beta y}e^{m\gamma z}$ .Substituting in the equation (13), we'll get

$$4q\psi(t+\Delta t)e^{m\propto x}e^{m\beta y}e^{m\gamma z} - 4\psi(t+\Delta t)e^{m\propto(x+\Delta x)}e^{m\beta y}e^{m\gamma z} + 8\psi(t+\Delta t)e^{m\propto x}e^{m\beta y}e^{m\gamma z} - 4\psi(t+ \Delta t em\propto x - \Delta x em\beta y em\gamma z = 4\psi t em\propto x em\beta y + \Delta y em\gamma z + 4\psi t em\propto x em\beta y - \Delta y em\gamma z + 4\psi t em\propto x em\beta y em\gamma z + \Delta z + 4 \psi(t)e^{m\propto x}e^{m\beta y}e^{m\gamma(z-\Delta z)} + \psi(t)e^{m\propto(x+\Delta x)}e^{m\beta(y+\Delta y)}e^{m\gamma z} - \psi(t)e^{m\propto(x+\Delta x)}e^{m\beta(y-\Delta y)}e^{m\gamma z} + \psi(t)e^{m\propto(x-\Delta x)}e^{m\beta(y-\Delta y)}e^{m\gamma z} - \psi(t)e^{m\propto(x-\Delta x)}e^{m\beta(y+\Delta y)}e^{m\gamma z} + 2q\psi(t)e^{m\propto(x+\Delta x)}e^{m\beta y}e^{m\gamma z} - 2q\psi(t)e^{m\propto(x-\Delta x)}e^{m\beta y}e^{m\gamma z} + 2q\psi(t)e^{m\propto x}e^{m\beta(y+\Delta y)}e^{m\gamma z} - 2q\psi(t)e^{m\propto x}e^{m\beta(y-\Delta y)}e^{m\gamma z} - 4q\psi(t)e^{m\propto x}e^{m\beta y}e^{m\gamma z} - 16\psi(t)e^{m\propto x}e^{m\beta y}e^{m\gamma z}). \quad (14)$$

Dividing equation (14) by $\psi(t)e^{m\propto x}e^{m\beta y}e^{m\gamma z}$ and factorizing will yield

$$|\lambda| = \frac{\psi(t+\Delta t)}{\psi t}(4q - 4e^{m\propto\Delta x} + 8 - 4e^{-m\propto\Delta x}) = 4e^{m\beta\Delta y} + 4e^{-m\beta\Delta y} + 4e^{m\gamma\Delta z} + 4e^{-m\gamma\Delta z} + 4\,Sin\left(\frac{\beta\Delta y}{2}\right)Cos\left(\frac{\beta\Delta y}{2}\right) - 8m\,Sin\left(\frac{\beta\Delta y}{2}\right)Cos\left(\frac{\beta\Delta y}{2}\right)Sin^2\left(\frac{\alpha\Delta x}{2}\right) + 2qe^{m\propto\Delta x} - 2qe^{-m\propto\Delta x} + 2qe^{m\beta\Delta y} - 2qe^{-m\beta\Delta y} - 4q - 16 \quad (15)$$

Equation (15) further simplifies to

$$|\lambda| = \frac{\psi(t+\Delta t)}{\psi t}[8 + 4q - (4e^{m\propto\Delta x} + 4e^{-m\propto\Delta x})] = 8\,Cos(\beta\Delta y) + 8\,Cos(\gamma\Delta z) + 8mSin\left(\frac{\beta\Delta y}{2}\right)Cos\left(\frac{\beta\Delta y}{2}\right)Sin^2\left(\frac{\alpha\Delta x}{2}\right) + 4qmSin(\propto\Delta x) + 4qmSin(\beta\Delta y) + 4q - 16 \quad (16)$$

Equation (16) reduces further to

$$|\lambda| = \frac{\psi(t+\Delta t)}{\psi t}[8 + 4q - 8\,Cos(\propto\Delta x)] = 8\,Cos(\beta\Delta y) + 8\,Cos(\gamma\Delta z) + 4mSin\left(\frac{\beta\Delta y}{2}\right)Cos\left(\frac{\beta\Delta y}{2}\right) - 8mSin\left(\frac{\beta\Delta y}{2}\right)Cos\left(\frac{\beta\Delta y}{2}\right)Sin^2\left(\frac{\alpha\Delta x}{2}\right) + 4qmSin(\propto\Delta x) + 4qmSin(\beta\Delta y) - 4q - 16 \quad (17)$$

We can write equation (17) as

$$|\lambda| = \left|\frac{\psi(t+\Delta t)}{\psi(t)}\right| = \left|\frac{8Cos(\beta\Delta y) + 8\,Cos(\gamma\Delta z) + 4mSin\left(\frac{\beta\Delta y}{2}\right) - 8mSin\left(\frac{\beta\Delta y}{2}\right)Cos\left(\frac{\beta\Delta y}{2}\right)Sin^2\left(\frac{\alpha\Delta x}{2}\right) + 4qmSin(\alpha\Delta x) + 4qmSin(\beta\Delta y) - 4q - 16}{8 + 4q - 8Cos(\propto\Delta x)}\right| \leq 1 \quad (18)$$

Equation (18) can now be written as

$$|\lambda| = \left|\frac{\psi(t+\Delta t)}{\psi t}\right| = \left|\frac{8 - 16\,Sin^2\left(\frac{\beta\Delta y}{2}\right) + 8 - 16\,Sin^2\left(\frac{\gamma\Delta z}{2}\right) + 4mSin\left(\frac{\beta\Delta y}{2}\right)Cos\left(\frac{\beta\Delta y}{2}\right) + -8mSin\left(\frac{\beta\Delta y}{2}\right)Cos\left(\frac{\beta\Delta y}{2}\right)Sin^2\left(\frac{\alpha\Delta x}{2}\right) + 8qmSin\left(\frac{\alpha\Delta x}{2}\right)Cos\left(\frac{\alpha\Delta x}{2}\right) + 8qmSin\left(\frac{\beta\Delta y}{2}\right)Cos\left(\frac{\beta\Delta y}{2}\right) - 4q - 16}{8 + 4q - 8 + 16\,Sin^2\left(\frac{\propto\Delta x}{2}\right)}\right| \leq 1 \quad (19)$$

We analyse the algorithm of (19). This is done for the maximum and minimum amplification factor λ

(a) For maximum value of λ we take $\left(\frac{\propto\Delta x}{2}\right), \left(\frac{\beta\Delta x}{2}\right), \left(\frac{\gamma\Delta x}{2}\right)$ to be equal to zero, hence

$$|\lambda| = \left|\frac{\psi(t+\Delta t)}{\psi(t)}\right| = \left|\frac{8+8-4q-16}{4q+16}\right| \leq 1 (20)$$

Equation (20) further reduces to

$$|\lambda| = \left|\frac{-(4q+16}{4q+16}\right| \leq 1,$$ (Denominator is greater than the Numerator).(21)

(b) For minimum value of λ we take $\left(\frac{\propto\Delta x}{2}\right), \left(\frac{\beta\Delta x}{2}\right), \left(\frac{\gamma\Delta x}{2}\right)$ to be equal to 90^0, hence

$$|\lambda| = \left|\frac{\psi(t+\Delta t)}{\psi(t)}\right| = \left|\frac{8-16+8-16-4q-16}{8+4q-8+16}\right| \leq 1 (22)$$

$$|\lambda| = \left|\frac{q-12}{q+4}\right| \leq 1,$$ (Denominator is greater than the Numerator).(23)

Conclusion

It can be seen clearly that $|\lambda|$ will always be less than one for both (3) and (13) respectively. All the values in (a) and (b) are less than one since the denominator is greater than the numerator. Thus the ADE and ADI schemes are stable for all values of p ie unconditionally stable.

References

[1] Ebach E.H and White, R. Mixing of fluids flowing through beds of packed soils – Journal of American Institute of Chemical Engineering, 4,(1958) 161-164.

[2] Elder, J. W. The dispersion of marked fluid in turbulent shear flow – Journal of Fluid Mechanics 5,(1956), 544-560.

[3] Ellsworth T. R and Butters G. L. Three-dimensional analytical solution to the advection-dispersion equation in arbitrary cartesian coordinate- Water Resource Research-20,(1993) 1337-1354.

[4] Essa, K. S. M, Etmas, S. M and Embaby,M.New analytical solution of the dispersion equation. Atmospheric Research, 84, (2007) 337-334.

[5] Fischer H. B. The mechanics of dispersion in natural streams. Journal of Hydraulic Division ASCE, 93, (1967) 187-216.

[6] George O.O., Ongati O. and Oduor O. E. M. Pure implicit Finite Difference Solution of a Parabolic Advection-Diffusion Equation with Constant Coefficients International Journal of Engineering and Mathematical Science. January-June, Volume 9,(2016) Issue-1.

[7] Ghosh, N.C., Sharma K.D.,Ground Water Modelling and Management, Capital Publishing Company, New Delhi (2006)

[8] Gottlieb D and Gustafsson B.Generalized Du Fort Franked Methods for parabolic, Volume 13, (1996), Number 1.

[9] Griths D. F. and Mitchel A. R. -The Finite Difference Methods in Partial Differential Equations. John Wiley and Sons,(1980).

[10] Gunn J.E. Alternating Direction Methods for three space variables. Numerical Techniques 4(2012) 41-63.

[11] Johnson, S. Saad, Y. and Schultz, M.,Alternating Direction Methods on Multiprocessors, SIAM, J.sci. Statist. Comput. 8,pp. 686-700 (1987).

[12] Lapidus, L. and Pinder, G.F., Numerical Solution of Partial Differential Equations in Science and Engineering, John Wiley & Sons Inc (1982).

[13] Larkin, B.K.,Some Stable Explicit Difference Approximations to the Diffusion Equation, South Grant Littleton, Colorado pp. 196-202 (1963).

[14] Ming-shu, M. and Tong-ke, W.,A family of High-order Accuracy Explicit Difference Schemes with Branching Stability for Solving 3-D Parabolic Partial Differential Equation, University Shanghai, China vol. 21, pp. 1207-1212 (2000).

[15] Morton, K.W. and Mayers, D.F.,Numerical Solution of Partial Differential Equations, Cambridge University Press, UK. Second ed (2005).

[16] Noye, B.J. and Hayman, K.J., New LOD and ADI Methods for The Two-Dimensional Diffusion Equation, J. computer Mathematics, vol. 51, pp. 215-228 (1994).

[17] Peaceman, D.W. and Rachford, H.H.,The Numerical Solution of Parabolic andElliptic Differential Equations,J.soc. Indust. Appl. Math. 3, pp. 28-41 (1955).