On The Homogeneous Ternary Quadraticdiophantine Equation $6x^2+7y^2=559z^2$

Dr. S.Mallika

Assistant professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-2.

ABSTRACT:

Homogeneous ternary quadratic equation $6x^2+7y^2=559z^2$ *is analysed for its integral points on it. Employing the integral solutions of the above equation, a few interesting relations between the solutions and the special numbers are also exhibited.*

KEYWORDS: *Homogeneous, ternary, quadratic, integral solutions.*

I. INTRODUCTION

The Diophantine equation offer an unlimited field for research due to their variety $[1-3]$. In particular, one may refer $[4-13]$ for quadratic equation with three unknowns. The communication concerns with yet another interesting equation $6x^2+7y^2=559z^2$ representing non-homogeneous quadratic equation with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

II. NOTATIONS

1. Polygonal number of rank n with side m

$$t_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$$

2. Pronic number of rank n

$$\Pr_n=n(n+1)$$

3. Star number of rank n

$$S_n=6n(n-1)+1.$$

III. METHOD OF ANALYSIS

The ternary quadratic Diophantine equation to be solved for its non- zero distinct integral solution is

$$6x^2+7y^2=559z^2 \tag{1}$$

Introduction of the linear transformations $(X\neq T\neq 0)$

$$x=X-7T,\ y=X+6T \tag{2}$$

in (1) leads to

$$X^2+42T^2=43z^2 \tag{3}$$

Different patterns of solutions of (1) are presented below:

Pattern-1:

Write 43 as

$$43=(1+i\sqrt{42})(1-i\sqrt{42}) \tag{4}$$

Assume

$$z=a^2+42b^2 \tag{5}$$

where a and b are non- zero distinct integers

using (4) and (5) in (3), we get

$$X^2+42T^2=(1+i\sqrt{42})(1-i\sqrt{42})(a^2+42b^2)^2$$

Employing the method of factorization the above equation is written as

$$\left(X+i\sqrt{42}T\right)\left(X-i\sqrt{42}T\right)=\left(1+i\sqrt{42}\right)\left(1-i\sqrt{42}\right)\left(a+i\sqrt{42}b\right)^2\left(a-i\sqrt{42}b\right)^2$$

Equating the positive and negative factors, the resulting equations are,

$$X+i\sqrt{42}T=\left(1+i\sqrt{42}\right)\left(a+i\sqrt{42}b\right)^2 \tag{6}$$

$$X-i\sqrt{42}T=\left(1-i\sqrt{42}\right)\left(a-i\sqrt{42}b\right)^2 \tag{7}$$

Equating real and imaginary parts in (6), we get

$$X=a^2-42b^2-84ab$$

$$T=a^2-42b^2+2ab$$

substituting the values of X and T in (2), we get

$$x=x(a,b)=-6a^2+252b^2-98ab \tag{8}$$

$$y=y(a,b)=7a^2-294b^2-72ab \tag{9}$$

Thus (8), (9) and (5) represents non- zero distinct integral solutions of (1) in two parameters.

Properties:

- $x(1,b)+98\,\mathrm{Pr}_b-154t_{4,b}\equiv 0(\mathrm{mod}\,2)$
- $y(1,b)+72\,\mathrm{Pr}_b+222t_{4,b}\equiv 0(\mathrm{mod}\,7)$
- $x(b,b)+y(b,b)+211t_{4,b}=0$
- $y(b,b)+z(b,b)+312t_{4,b}=0$
- $x(a+1,1)+S_a+116\,\mathrm{Pr}_a-116t_{4,a}\equiv 0(\mathrm{mod}\,2)$
- $z(1,b+1)-S_b-126\,\mathrm{Pr}_b+126t_{4,b}\equiv 1(\mathrm{mod}\,7)$

Pattern-2:

The equation (3) can also be written as

$$X^2-z^2=42\left(z^2-T^2\right)$$

$$(X-z)(X+z)=42(z-T)(z+T) \tag{10}$$

Equation (10) is written in the form of ratio as,

$$\frac{X+z}{2(z+T)}=\frac{21(z-T)}{X-z}=\frac{\alpha}{\beta},\beta\neq 0 \qquad (11)$$

From the first and third factors of (11), we have

$$\frac{X+z}{2(z+T)}=\frac{\alpha}{\beta}$$

$$\beta X-2\alpha T+(\beta-2\alpha)z=0 \qquad (12)$$

From the second and third factors of (12), we have

$$\frac{21(z-T)}{X-z}=\frac{\alpha}{\beta}$$

$$-\alpha X-21\beta T+(21\beta+\alpha)z=0 \qquad (13)$$

Applying the method of cross multiplication for solving (12) and (13)

$$X=2\alpha^2+84\alpha\beta-21\beta^2$$

$$T=-2\alpha^2+2\alpha\beta+21\beta^2$$

$$z=2\alpha^2+21\beta^2$$

substituting the values of X and T in (2), we get

$$\left.\begin{aligned} x=x(\alpha,\beta)&=-12\alpha^2+98\alpha\beta-168\beta^2\\ y=y(\alpha,\beta)&=-10\alpha^2+96\alpha\beta+105\beta^2\\ z=z(\alpha,\beta)&=2\beta^2+21\alpha^2 \end{aligned}\right\} \qquad (14)$$

Thus (14) represents the non-zero distinct integral solutions to (1)

Properties:

- $x(1,\beta)-98\Pr_\beta+266t_{4,\beta}\equiv 0(\bmod 2)$
- $y(1,\beta)-96\Pr_\beta-9t_{4,\beta}\equiv 0(\bmod 5)$
- $x(\beta,\beta)+y(\beta,\beta)+109t_{4,\beta}=0$
- $y(\beta,\beta)+z(\beta,\beta)-214t_{4,\beta}=0$
- $x(\alpha+1,1)+S_\alpha-64\Pr_\alpha+64t_{4,\alpha}\equiv 0(\bmod 9)$
- $z(1,\beta+1)-S_\beta-63\Pr_\beta+63t_{4,\beta}\equiv 0(\bmod 2)$

Pattern-3:

One may write (10) in the form of ratio as

$$\frac{X+z}{21(z+T)}=\frac{2(Z-T)}{X-z}=\frac{\alpha}{\beta},\beta\neq 0 \qquad (15)$$

From the first and third factors of (15), we have

$$\frac{X+z}{21(z+T)}=\frac{\alpha}{\beta}$$

$$\beta X+(\beta-21\alpha)z-21\alpha T=0 \qquad (16)$$

From the second and third factors of (15), we have

$$\frac{2(z-T)}{X-z}=\frac{\alpha}{\beta}$$

$$-2\beta T+(2\beta+\alpha)z-\alpha X=0 \tag{17}$$

Applying the method of cross multiplication for solving (16) and (17)

$$X=21\alpha^2+84\alpha\beta-2\beta^2$$
$$T=2\beta^2+2\alpha\beta-21\alpha^2$$
$$z=21\alpha^2+2\beta^2$$

substituting the values of X and T in (2), we get

$$\left.\begin{aligned} x=x(\alpha,\beta)&=-16\beta^2+70\alpha\beta+168\alpha^2\\ y=y(\alpha,\beta)&=10\beta^2+96\alpha\beta-105\alpha^2\\ z=z(\alpha,\beta)&=21\alpha^2+2\beta^2 \end{aligned}\right\} \tag{18}$$

Thus (18) represents the non-zero distinct integral solutions to (1)

Properties:

- $x(1,\beta)-70\Pr_\beta+86t_{4,\beta}\equiv 0 \pmod 2$
- $y(1,\beta)-96\Pr_\beta+86t_{4,\beta}\equiv 0 \pmod 5$
- $x(\beta,\beta)+y(\beta,\beta)-223t_{4,\beta}=0$
- $y(\beta,\beta)+z(\beta,\beta)-24t_{4,\beta}=0$
- $x(\alpha+1,1)+S_\alpha-86\Pr_\alpha+86t_{4,\alpha}\equiv 0 \pmod 2$
- $z(1,\beta+1)-S_\beta-6\Pr_\beta+6t_{4,\beta}\equiv 0 \pmod 2$

Pattern-4:

One may write (10) in the form of ratio as

$$\frac{X+z}{21(z-T)}=\frac{2(Z+T)}{X-z}=\frac{\alpha}{\beta},\beta\neq 0 \tag{19}$$

From the first and third factors of (19), we have

$$\frac{X+z}{21(z-T)}=\frac{\alpha}{\beta}$$

$$\beta X+(\beta-21\alpha)z+21\alpha T=0 \tag{20}$$

From the second and third factors of (19), we have

$$\frac{2(z+T)}{X-z}=\frac{\alpha}{\beta}$$

$$2\beta T+(2\beta+\alpha)z-\alpha X=0 \tag{21}$$

Applying the method of cross multiplication for solving (20) and (21)

$$X=-21\alpha^2-84\alpha\beta+2\beta^2$$
$$T=2\beta^2+2\alpha\beta-21\alpha^2$$
$$z=-21\alpha^2-2\beta^2$$

substituting the values of X and T in (2), we get

$$\left.\begin{aligned} x=x(\alpha,\beta)&=-12\beta^2-98\alpha\beta+126\alpha^2\\ y=y(\alpha,\beta)&=14\beta^2-72\alpha\beta-147\alpha^2\\ z=z(\alpha,\beta)&=-2\alpha^2-21\beta^2 \end{aligned}\right\} \tag{22}$$

Thus (22) represents the non-zero distinct integral solutions to (1)

Properties:

- $x(1,\beta)+98\,\mathrm{Pr}_{\beta}-86t_{4,\beta}\equiv 0(\bmod 3)$
- $y(1,\beta)+72\,\mathrm{Pr}_{\beta}-86t_{4,\beta}\equiv 0(\bmod 7)$
- $x(\beta,\beta)+y(\beta,\beta)+189t_{4,\beta}=0$
- $y(\beta,\beta)+z(\beta,\beta)+228t_{4,\beta}=0$
- $x(\alpha+1,1)-S_{\alpha}-280\,\mathrm{Pr}_{\alpha}+280t_{4,\alpha}\equiv 0(\bmod 5)$
- $z(1,\beta+1)+S_{\beta}+6\,\mathrm{Pr}_{\beta}-6t_{4,\beta}\equiv 0(\bmod 2)$

Pattern-5:

One may write (10) in the form of ratio as

$$\frac{X+z}{14(z+T)}=\frac{3(Z-T)}{X-z}=\frac{\alpha}{\beta},\beta\neq 0 \tag{23}$$

From the first and third factors of (23), we have

$$\frac{X+z}{14(z+T)}=\frac{\alpha}{\beta}$$

$$\beta X+(\beta-14\alpha)z-14\alpha T=0 \tag{24}$$

From the second and third factors of (24), we have

$$\frac{3(z-T)}{X-z}=\frac{\alpha}{\beta}$$

$$-3\beta T+(3\beta+\alpha)z-\alpha X=0 \tag{25}$$

Applying the method of cross multiplication for solving (24) and (25)

$$X=14\alpha^2+84\alpha\beta-3\beta^2$$

$$T=3\beta^2+2\alpha\beta-14\alpha^2$$

$$z=14\alpha^2+3\beta^2$$

substituting the values of X and T in (2), we get

$$\left.\begin{array}{l} x=x(\alpha,\beta)=-24\beta^2+70\alpha\beta+112\alpha^2 \\ y=y(\alpha,\beta)=15\beta^2+96\alpha\beta-70\alpha^2 \\ z=z(\alpha,\beta)=14\alpha^2+3\beta^2 \end{array}\right\} \tag{26}$$

Thus (26) represents the non-zero distinct integral solutions to (1)

Properties:

- $x(1,\beta)+-70\,\mathrm{Pr}_{\beta}+94t_{4,\beta}\equiv 0(\bmod 2)$
- $y(1,\beta)-96\,\mathrm{Pr}_{\beta}+81t_{4,\beta}\equiv 0(\bmod 2)$
- $x(\beta,\beta)+y(\beta,\beta)-199t_{4,\beta}=0$
- $y(\beta,\beta)+z(\beta,\beta)-58t_{4,\beta}=0$
- $x(\alpha+1,1)-S_{\alpha}-406\,\mathrm{Pr}_{\alpha}+406t_{4,\alpha}\equiv 1(\bmod 2)$
- $z(1,\beta+1)-S_{\beta}-9\,\mathrm{Pr}_{\beta}+9t_{4,\beta}\equiv 0(\bmod 2)$

Pattern-6:

One may write (10) in the form of ratio as

$$\frac{X+z}{3(z+T)}=\frac{14(Z-T)}{X-z}=\frac{\alpha}{\beta},\beta\neq 0 \qquad (27)$$

From the first and third factors of (27), we have

$$\frac{X+z}{3(z+T)}=\frac{\alpha}{\beta}$$

$$\beta X+(\beta-3\alpha)z-3\alpha T=0 \qquad (28)$$

From the second and third factors of (27), we have

$$\frac{14(z-T)}{X-z}=\frac{\alpha}{\beta}$$

$$-14\beta T+(14\beta+\alpha)z-\alpha X=0 \qquad (29)$$

Applying the method of cross multiplication for solving (28) and (29)

$$X=-14\beta^2+84\alpha\beta+3\alpha^2$$

$$T=14\beta^2+2\alpha\beta-3\alpha^2$$

$$z=3\alpha^2+14\beta^2$$

substituting the values of X and T in (2), we get

$$\left.\begin{aligned} x&=x(\alpha,\beta)=-112\beta^2+70\alpha\beta+24\alpha^2\\ y&=y(\alpha,\beta)=70\beta^2+96\alpha\beta-15\alpha^2\\ z&=z(\alpha,\beta)=3\alpha^2+14\beta^2 \end{aligned}\right\} \qquad (30)$$

Thus (30) represents the non-zero distinct integral solutions to (1)

Properties:

- $x(1,\beta)-70\,\mathrm{Pr}_\beta+182t_{4,\beta}\equiv 0(\mathrm{mod}\,2)$
- $y(1,\beta)-96\,\mathrm{Pr}_\beta+26t_{4,\beta}\equiv 0(\mathrm{mod}\,5)$
- $x(\beta,\beta)+y(\beta,\beta)-133t_{4,\beta}=0$
- $y(\beta,\beta)+z(\beta,\beta)-168t_{4,\beta}=0$
- $x(\alpha+1,1)-S_\alpha-406\,\mathrm{Pr}_\alpha+406t_{4,\alpha}\equiv 1(\mathrm{mod}\,2)$
- $z(1,\beta+1)-S_\beta-42\,\mathrm{Pr}_\beta+42t_{4,\beta}\equiv 0(\mathrm{mod}\,2)$

Pattern-7:

(10) may be written in the form of ratio as

$$\frac{X+z}{3(z-T)}=\frac{14(Z+T)}{X-z}=\frac{\alpha}{\beta},\beta\neq 0 \qquad (31)$$

From the first and third factors of (31), we have

$$\frac{X+z}{3(z-T)}=\frac{\alpha}{\beta}$$

$$\beta X+(\beta-3\alpha)z+3\alpha T=0 \qquad (32)$$

From the second and third factors of (32), we have

$$\frac{14(z+T)}{X-z}=\frac{\alpha}{\beta}$$

$$14\beta T+(14\beta+\alpha)z-\alpha X=0 \quad (33)$$

Applying the method of cross multiplication for solving (32) and (33)

$$X=-3\alpha^2-84\alpha\beta+14\beta^2$$
$$T=14\beta^2+2\alpha\beta-3\alpha^2$$
$$z=-3\alpha^2-14\beta^2$$

substituting the values of X and T in (2), we get

$$\left.\begin{aligned} x=x(\alpha,\beta)&=-84\beta^2-98\alpha\beta+18\alpha^2\\ y=y(\alpha,\beta)&=98\beta^2-72\alpha\beta-21\alpha^2\\ z=z(\alpha,\beta)&=-3\alpha^2-14\beta^2\end{aligned}\right\} \quad (34)$$

Thus (34) represents the non-zero distinct integral solutions to (1)

Properties:

- $x(1,\beta)+98\Pr_\beta-14t_{4,\beta}\equiv 0(\mathrm{mod}\,2)$
- $y(1,\beta)+72\Pr_\beta-170t_{4,\beta}\equiv 0(\mathrm{mod}\,3)$
- $x(\beta,\beta)+y(\beta,\beta)+159t_{4,\beta}=0$
- $y(\beta,\beta)+z(\beta,\beta)+12t_{4,\beta}=0$
- $x(\alpha+1,1)-S_\alpha+44\Pr_\alpha-44t_{4,\alpha}\equiv 0(\mathrm{mod}\,5)$
- $z(1,\beta+1)+S_\beta+42\Pr_\beta-42t_{4,\beta}\equiv 0(\mathrm{mod}\,2)$

Pattern-8:

One may write (10)in the form of ratio as

$$\frac{X+z}{6(z+T)}=\frac{7(Z-T)}{X-z}=\frac{\alpha}{\beta},\beta\neq 0 \quad (35)$$

From the first and third factors of (35), we have

$$\frac{X+z}{6(z+T)}=\frac{\alpha}{\beta}$$

$$\beta X+(\beta-6\alpha)z-6\alpha T=0 \quad (36)$$

From the second and third factors of (35), we have

$$\frac{7(z-T)}{X-z}=\frac{\alpha}{\beta}$$

$$-7\beta T+(7\beta+\alpha)z-\alpha X=0 \quad (37)$$

Applying the method of cross multiplication for solving (36) and (37)

$$X=6\alpha^2+84\alpha\beta-7\beta^2$$
$$T=7\beta^2+2\alpha\beta-6\alpha^2$$
$$z=6\alpha^2+7\beta^2$$

substituting the values of X and T in (2), we get

$$\left.\begin{array}{l} x = x(\alpha,\beta) = -56\beta^2 + 70\alpha\beta + 48\alpha^2 \\ y = y(\alpha,\beta) = 35\beta^2 + 96\alpha\beta - 30\alpha^2 \\ z = z(\alpha,\beta) = 6\alpha^2 + 7\beta^2 \end{array}\right\} \quad (38)$$

Thus (38) represents the non-zero distinct integral solutions to (1)

Properties:

- $x(1,\beta) - 70\,\mathrm{Pr}_\beta + 126t_{4,\beta} \equiv 0 \pmod 2$
- $y(1,\beta) - 96\,\mathrm{Pr}_\beta + 61t_{4,\beta} \equiv 0 \pmod 5$
- $x(\beta,\beta) + y(\beta,\beta) - 163t_{4,\beta} = 0$
- $y(\beta,\beta) + z(\beta,\beta) - 114t_{4,\beta} = 0$
- $x(\alpha+1,1) - S_\alpha - 214\,\mathrm{Pr}_\alpha + 214t_{4,\alpha} \equiv 0 \pmod 2$
- $z(1,\beta+1) - S_\beta - 21\,\mathrm{Pr}_\beta + 21t_{4,\beta} \equiv 1 \pmod 2$

Pattern-9:

One may write (10) in the form of ratio as

$$\frac{X+z}{7(z+T)} = \frac{6(Z-T)}{X-z} = \frac{\alpha}{\beta}, \beta \neq 0 \quad (39)$$

From the first and third factors of (38), we have

$$\frac{X+z}{7(z+T)} = \frac{\alpha}{\beta}$$

$$\beta X + (\beta - 7\alpha)z - 7\alpha T = 0 \quad (40)$$

From the second and third factors of (40), we have

$$\frac{6(z-T)}{X-z} = \frac{\alpha}{\beta}$$

$$-6\beta T + (6\beta + \alpha)z - \alpha X = 0 \quad (41)$$

Applying the method of cross multiplication for solving (40) and (41)

$$X = 7\alpha^2 + 84\alpha\beta - 6\beta^2$$

$$T = 6\beta^2 + 2\alpha\beta - 7\alpha^2$$

$$z = 7\alpha^2 + 6\beta^2$$

substituting the values of X and T in (2), we get

$$\left.\begin{array}{l} x = x(\alpha,\beta) = -48\beta^2 + 70\alpha\beta + 56\alpha^2 \\ y = y(\alpha,\beta) = 30\beta^2 + 96\alpha\beta - 35\alpha^2 \\ z = z(\alpha,\beta) = 7\alpha^2 + 6\beta^2 \end{array}\right\} \quad (42)$$

Thus (42) represents the non-zero distinct integral solutions to (1)

Properties:

- $x(1,\beta) - 70\,\mathrm{Pr}_\beta + 118t_{4,\beta} \equiv 0 \pmod 2$
- $y(1,\beta) - 96\,\mathrm{Pr}_\beta + 66t_{4,\beta} \equiv 0 \pmod 5$
- $x(\beta,\beta) + y(\beta,\beta) - 169t_{4,\beta} = 0$

- $y(\beta,\beta)+z(\beta,\beta)-104t_{4,\beta}=0$
- $x(\alpha+1,1)-S_{\alpha}-238\Pr_{\alpha}+238t_{4,\alpha}\equiv 0(\bmod 2)$
- $z(1,\beta+1)-S_{\beta}-18\Pr_{\beta}+18t_{4,\beta}\equiv 1(\bmod 2)$

Pattern-10:

One may write (10) in the form of ratio as

$$\frac{X+z}{7(z-T)}=\frac{6(z+T)}{X-z}=\frac{\alpha}{\beta},\beta\neq 0 \tag{43}$$

From the first and third factors of (43), we have

$$\frac{X+z}{7(z-T)}=\frac{\alpha}{\beta}$$

$$\beta X+(\beta-7\alpha)z+7\alpha T=0 \tag{44}$$

From the second and third factors of (43), we have

$$\frac{6(z+T)}{X-z}=\frac{\alpha}{\beta}$$

$$6\beta T+(6\beta+\alpha)z-\alpha X=0 \tag{45}$$

Applying the method of cross multiplication for solving (44) and (45)

$$X=-7\alpha^2-84\alpha\beta+6\beta^2$$

$$T=6\beta^2+2\alpha\beta-7\alpha^2$$

$$z=-7\alpha^2-6\beta^2$$

substituting the values of X and T in (2), we get

$$\left.\begin{aligned} x&=x(\alpha,\beta)=-36\beta^2-98\alpha\beta+42\alpha^2\\ y&=y(\alpha,\beta)=42\beta^2-72\alpha\beta-49\alpha^2\\ z&=z(\alpha,\beta)=-7\alpha^2-6\beta^2\end{aligned}\right\} \tag{46}$$

Thus (46) represents the non-zero distinct integral solutions to (1)

Properties:

- $x(1,\beta)+98\Pr_{\beta}-62t_{4,\beta}\equiv 0(\bmod 2)$
- $y(1,\beta)+72\Pr_{\beta}-114t_{4,\beta}\equiv 1(\bmod 2)$
- $x(\beta,\beta)+y(\beta,\beta)+171t_{4,\beta}=0$
- $y(\beta,\beta)+z(\beta,\beta)+92t_{4,\beta}=0$
- $x(\alpha+1,1)-S_{\alpha}-28\Pr_{\alpha}+28t_{4,\alpha}\equiv 0(\bmod 2)$
- $z(1,\beta+1)+S_{\beta}+18\Pr_{\beta}-18t_{4,\beta}\equiv 1(\bmod 2)$

Pattern-11:

One may write (10) in the form of ratio as

$$\frac{X-z}{21(z+T)}=\frac{2(z-T)}{X+z}=\frac{\alpha}{\beta},\beta\neq 0 \tag{47}$$

From the first and third factors of (47), we have

$$\frac{X-z}{21(z+T)}=\frac{\alpha}{\beta}$$

$$\beta X-(\beta+21\alpha)z-21\alpha T=0 \tag{48}$$

From the second and third factors of (47), we have

$$\frac{2(z-T)}{X+z}=\frac{\alpha}{\beta}$$

$$-2\beta T+(2\beta-\alpha)z-\alpha X=0 \tag{49}$$

Applying the method of cross multiplication for solving (48) and (49)

$$X=-21\alpha^2+84\alpha\beta+2\beta^2$$

$$T=2\beta^2-2\alpha\beta-21\alpha^2$$

$$z=21\alpha^2+2\beta^2$$

substituting the values of X and T in (2), we get

$$\left.\begin{array}{l} x=x(\alpha,\beta)=-12\beta^2+98\alpha\beta+126\alpha^2 \\ y=y(\alpha,\beta)=14\beta^2+72\alpha\beta-147\alpha^2 \\ z=z(\alpha,\beta)=21\alpha^2+2\beta^2 \end{array}\right\} \tag{50}$$

Thus (50) represents the non-zero distinct integral solutions to (1)

Properties:

- $x(1,\beta)-98\Pr_\beta+110t_{4,\beta}\equiv 0(\mathrm{mod}\,2)$
- $y(1,\beta)-72\Pr_\beta+58t_{4,\beta}\equiv 0(\mathrm{mod}\,7)$
- $x(\beta,\beta)+y(\beta,\beta)-151t_{4,\beta}=0$
- $y(\beta,\beta)+z(\beta,\beta)+38t_{4,\beta}=0$
- $x(\alpha+1,1)-S_\alpha-476\Pr_\alpha+476t_{4,\alpha}\equiv 0(\mathrm{mod}\,2)$
- $z(1,\beta+1)-S_\beta-6\Pr_\beta+6t_{4,\beta}\equiv 1(\mathrm{mod}\,2)$

Pattern-12:

One may write (10) in the form of ratio as

$$\frac{X-z}{3(z+T)}=\frac{14(z-T)}{X+z}=\frac{\alpha}{\beta},\beta\neq 0 \tag{51}$$

From the first and third factors of (51), we have

$$\frac{X-z}{3(z+T)}=\frac{\alpha}{\beta}$$

$$\beta X-(\beta+3\alpha)z-3\alpha T=0 \tag{52}$$

From the second and third factors of (50), we have

$$\frac{14(z-T)}{X+z}=\frac{\alpha}{\beta}$$

$$-14\beta T+(14\beta-\alpha)z-\alpha X=0 \tag{53}$$

Applying the method of cross multiplication for solving (52) and (53)

$$X=-3\alpha^2+84\alpha\beta+14\beta^2$$

$$T=14\beta^2-2\alpha\beta-3\alpha^2$$

$$z = 3\alpha^2 + 14\beta^2$$

substituting the values of X and T in (2), we get

$$\left.\begin{aligned} x &= x(\alpha,\beta) = -84\beta^2 + 98\alpha\beta + 18\alpha^2 \\ y &= y(\alpha,\beta) = 98\beta^2 + 72\alpha\beta - 21\alpha^2 \\ z &= z(\alpha,\beta) = 3\alpha^2 + 14\beta^2 \end{aligned}\right\} \quad (54)$$

Thus (54) represents the non-zero distinct integral solutions to (1)

Properties:

- $x(1,\beta) - 98\Pr_\beta + 182t_{4,\beta} \equiv 0(\bmod 2)$
- $y(1,\beta) - 72\Pr_\beta + 170t_{4,\beta} \equiv 0(\bmod 3)$
- $x(\beta,\beta) + y(\beta,\beta) - 181t_{4,\beta} = 0$
- $y(\beta,\beta) + z(\beta,\beta) - 166t_{4,\beta} = 0$
- $x(\alpha+1,1) - S_\alpha - 152\Pr_\alpha + 152t_{4,\alpha} \equiv 0(\bmod 2)$
- $z(1,\beta+1) - S_\beta - 42\Pr_\beta + 42t_{4,\beta} \equiv 1(\bmod 2)$

Pattern-13:

One may write (10) in the form of ratio as

$$\frac{X - z}{7(z+T)} = \frac{6(z-T)}{X+z} = \frac{\alpha}{\beta}, \beta \neq 0 \quad (55)$$

From the first and third factors of (55), we have

$$\frac{X - z}{7(z+T)} = \frac{\alpha}{\beta}$$

$$\beta X - (\beta + 7\alpha)z - 7\alpha T = 0 \quad (56)$$

From the second and third factors of (56), we have

$$\frac{6(z-T)}{X+z} = \frac{\alpha}{\beta}$$

$$-6\beta T + (6\beta - \alpha)z - \alpha X = 0 \quad (57)$$

Applying the method of cross multiplication for solving (56) and (57)

$$X = -7\alpha^2 + 84\alpha\beta + 6\beta^2$$

$$T = 6\beta^2 - 2\alpha\beta - 7\alpha^2$$

$$z = 7\alpha^2 + 6\beta^2$$

substituting the values of X and T in (2), we get

$$\left.\begin{aligned} x &= x(\alpha,\beta) = -36\beta^2 + 98\alpha\beta + 42\alpha^2 \\ y &= y(\alpha,\beta) = 42\beta^2 + 72\alpha\beta - 49\alpha^2 \\ z &= z(\alpha,\beta) = 7\alpha^2 + 6\beta^2 \end{aligned}\right\} \quad (58)$$

Thus (58) represents the non-zero distinct integral solutions to (1)

Properties:

- $x(1,\beta) - 98\Pr_\beta + 134t_{4,\beta} \equiv 0(\bmod 2)$

- $y(1,\beta)-72\Pr_{\beta}+30t_{4,\beta}\equiv 0(\bmod 7)$
- $x(\beta,\beta)+y(\beta,\beta)-169t_{4,\beta}=0$
- $y(\beta,\beta)+z(\beta,\beta)-78t_{4,\beta}=0$
- $x(\alpha+1,1)-S_{\alpha}-224\Pr_{\alpha}+224t_{4,\alpha}\equiv 0(\bmod 2)$
- $z(1,\beta+1)-S_{\beta}-18\Pr_{\beta}+18t_{4,\beta}\equiv 1(\bmod 2)$

IV. CONCLUSION

In this paper, we have made an attempt to obtain all integer solutions to ternary quadratic equation $6x^2+7y^2=559z^2$.One may search for other patterns of solutions and their corresponding properties.

V. REFERENCES

[1] Dickson L.E. "History of theory of numbers", Vol-2, Diophantine analysis, New York, Dover, 2005.

[2] Mordell L.J. "Diophantine equations", Academic press, New York, 1969.

[3] Carmichael R.D. "The theory of numbers and Diophantine analysis", New York, Dover, 1959.

[4] Gopalan M.A, Geetha D, Lattice points on the hyperbola of two sheets $x^2-6xy+y^2+6x-2y+5=z^2+4$, Impact J sci Tech : 4:23-32,2010.

[5] Gopalan M.A, Vidhyalakshmi S, Kavitha A, Integral points on the homogeneous cone $z^2=2x^2-7y^2$, The Diophantine J Math ;$1(2):127-136,2012.$.

[6] Gopalan M.A, Vidhyalakshmi S, Sumathi G, Lattice points on the hyperboloid of one sheet $4z^2=2x^2+3y^2-4$,Diophantine J Math;$1(2):109-115,2012.$.

[7] Gopalan M.A, Vidhyalakshmi S, Lakshmi K, Integral points on the hyperboloid of two sheets $3y^2=7x^2-z^2+21$, Diophantine J Math ;$1(2):99-107,2012.$

[8] Gopalan M.A, Vidhyalakshmi S, Mallika S, Observations on hyperboloid of one sheet $x^2+2y^2-z^2=2$, Bessel J Math ; $2(3):221-226,2012.$.

[9] Gopalan M.A, Vidhyalakshmi S, Usha Rani T.R, Mallika S, Integral points on the homogeneous cone $6z^2+3y^2-2x^2=0$, Impact J sci Tech ; $6(1):7-13,2012.$.

[10] Gopalan M.A, Vidhyalakshmi S, Sumathi G, Lattice points on the elliptic paraboloid $z=9x^2+4y^2$, Advances in Theoretical and Applied Mathematics;m $7(4):379-385,2012.$.

[11] Gopalan M.A, Vidhyalakshmi S, Usha Rani T.R, Integral points on the non-homogeneous cone $2z^2+4xy+8x-4z=0$, Global Journal of Mathematics and Mathematics Sciences 2012; $2(1):61-67$.

[12] Gopalan M.A, Vidhyalakshmi S, Lakshmi K, Lattice points on the elliptic paraboloid $16y^2+9z^2=4x$, Bessel J of Math ;$_s$

[13] Gopalan M.A, Geetha T, HemalathaK,"On the ternary quadratic Diophantine equation $5(x^2+y^2)-2xy=20z^2$", International journal of multidisciplinary research and development, Vol-2, Issue: 4,211-214, April-2015.

[14] Hema.D,Mallika.S,"On the ternary quadratic Diophantine equation $5y^2=3x^2+2z^2$,Journal of Mathematics and informatics,Vol.10,157-165,2017.

[15] Selva Keerthana.K , Mallika.S, "On the Ternary Quadratic Diophantine equation $3(x^2+y^2)-5xy+2(x+y)+4=15z^2$"Journal of Mathematics and informatics,Vol. 11,21-28,2017.