Oscillations of Fourth Order Linear Neutral Delay Differential Equations

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Abstract: Sufficient conditions for oscillations of fourth order linear neutral delay differential equations of the form

$$\frac{d}{dt}\left\{r(t)\frac{d^3}{dt^3}\left(m(t)y(t)+\sum_{i=1}^{n}\frac{1}{t(t-1)}y^{\alpha}(t-\tau)\right)\right\}+f(t)y^{\alpha}(t-\sigma)=0,\quad t\geq t_0$$

are obtained ,where , $r(t), m(t)$ are positive real valued continuous functions $f(t)\geq 0$, and α is the ratio of odd positive integers and n is an integer.

Key words: Oscillation, Fourth order, Neutral Differential equation.

I. INTRODUCTION

In this paper we consider the linear neutral delay differential equation

$$\frac{d}{dt}\left\{r(t)\frac{d^3}{dt^3}\left(m(t)y(t)+\sum_{i=1}^{n}\frac{1}{t(t-1)}y^{\alpha}(t-\tau)\right)\right\}+f(t)y^{\alpha}(t-\sigma)=0,\quad t\geq t_0 \quad (1)$$

where $r(t)\in C([t_0,\infty),(0,\infty))$, $f(t)\in C([t_0,\infty),[0,\infty))$.

Corresponding equation in the absence of neutral term is given by

.

$$\frac{d}{dt}\left\{r(t)\frac{d^3}{dt^3}\{m(t)y(t)\}\right\}+f(t)y^{\alpha}(t-\sigma))=0 \quad (2)$$

which is a delay differential equation and further if we take $m(t)=1, \sigma=0$ in equation (2) we get

$$\frac{d}{dt}\left\{r(t)\frac{d^3}{dt^3}\{y(t)\}\right\}+f(t)y^{\alpha}(t))=0 \quad (3)$$

The study of behavior of solutions of differential equation (2) has been a subject of interest for several researchers. We mention the works of [13, 2, 6 and 5].Oscillatory behavior of delay differential equations is extensively studied by several authors [7, 8, 9, 14, 4, 15 and 16].

Now we see some special case of equation (1).When

$r(t)\equiv 1$ equation (1) is reduced to

$$\frac{d^4}{dt^4}\left\{m(t)y(t)+\sum_{i=1}^{n}\frac{1}{t(t-1)}y^{\alpha}(t-\tau)\right\}+f(t)y^{\alpha}(t-\sigma)=0 \tag{4}$$

and to

$$\frac{d^4}{dt^4}\left\{m(t)y(t)+\sum_{i=1}^{n}\frac{1}{t(t-1)}y^{\alpha}(t-\tau)\right\}+f(t)y(t)=0 \text{ if } \quad \sigma=0 \tag{5}$$

and we note that, when $m(t)=1$, this equation further becomes to the equation

$$\frac{d^4}{dt^4}\left\{y(t)+\sum_{i=1}^{n}\frac{1}{t(t-1)}y^{\alpha}(t-\tau)\right\}+f(t)y(t)=0 \tag{6}$$

Recently there has been an increasing interest in the study of the oscillation of differential equations e.g. papers [1]-[12]. In particular, differential equations of the form (1) and for special cases when $r(t)\equiv 1$, is a subject of intensive research.

The oscillation for equation (6) has been discussed by many authors.

Said R. Grace , Jozef Džurina , Irena Jadlovská and Tongxing Li [14],studied the oscillatory behavior of the fourth order nonlinear differential equation

$$\left(r_3\left(r_2\left(r_1 y'\right)'\right)'\right)'(t)+q(t)y(\tau(t))=0 \tag{7}$$

and Jozef Džurina, Blanka Baculíková and Irena Jadlovská B [5] have considered the fourth order nonlinear neutral differential equation of the form

$$\left(r_3(t)\left(r_2(t)\left(r_1(t)y(t)'\right)'\right)'\right)'+p(t)y'(t)+q(t)y(\tau(t))=0 \tag{8}$$

Parhi and Tripathy [12] have considered fourth order neutral differential equation of the form

$$\left[r(t)\left(y(t)+p(t)y(t-\tau)\right)''\right]''+q(t)G(y(t-\sigma))=f(t)$$

and they have established the oscillation and asymptotic behavior of the equation under the condition

$$\int_{t_0}^{t} \frac{t}{r(t)} dt < \infty \text{ as } \quad t \to \infty$$

and
$$\int_{t_0}^{t} \frac{t}{r(t)} dt = \infty \text{ as } \quad t \to \infty \tag{9}$$

The present work is motivated by [13] where the Authors, P. V. H. S Sai Kumar and K. V. V Seshagiri Rao have considered oscillations of third order linear neutral delay differential equation of the form

$$\frac{d}{dt}\left\{r_1(t)\frac{d^2}{dt^2}\left(m(t)y(t)+\frac{r(t)}{r(t-\tau)}y^{\alpha}(t-\tau)\right)\right\}+f(t)y(t-\sigma)=0\,; \qquad t \ge t_0$$

In this paper we establish the conditions for the oscillation of solutions of equation (1) by Ricccati Technique using the condition.

$$\int_{t_0}^{t} \frac{1}{r(t)} dt = \infty \text{ as } \quad t \to \infty\,.$$

By a solution of equation (1) we mean a function $y(t) \in C([T_y, \infty))$ where $T_y \ge t_0$ which satisfies(1) on $[T_y, \infty)$. We consider only those solutions of *y(t)* of (1) which satisfy $Sup\{|y(t)| : t \ge T\} > 0$ for all $T \ge T_y$ and assume that (1) possesses such solutions.

A solution of equation (1) is called oscillatory if it has arbitrary large zeros on $[T_y, \infty)$; otherwise it is called nonoscillatory. Equation (1) is said to be oscillatory it all its solutions oscillate .Unless otherwise stated, when we write a functional inequality, it will be assumed to hold for sufficiently large t in our subsequent discussion.

II. MAIN RESULTS

We need the following in our discussion

(H_1): $r(t), m(t), \in C([t_0, \infty), R)$;

(H_2): $f(t)$, $p(t) = \dfrac{1}{t(t-1)}$ are continuously differentiable on $[t_0, \infty)$.

(H_3): $0 < \alpha \le 1$, and α is the ratio of odd positive integers.

(H_4): $\tau \in C'([t_0, \infty), R)$ and $\sigma \in C'([t_0, \infty), R)$.

(H_5): $f(t) > 0$, $0 \le p(t) < \infty$ for $i = 1, 2 \ldots\ldots\ldots \infty$.

We set

$$z(t)=m(t)y(t)+\sum_{i=1}^{n}\frac{1}{t(t-1)}y^{\alpha}(t-\tau) \tag{10}$$

and

$$R(t)=\int_{t0}^{t}\frac{1}{r^{\frac{1}{\alpha}}(t)}dt=\infty \text{ as } t\to\infty \tag{11}$$

We have the following Lemmas

Lemma 2.1: If X and Y are nonnegative and $\lambda > 1$, then

$$\lambda XY^{\lambda-1}-X^{\lambda}\leq(\lambda-1)Y^{\lambda},$$

where equality holds if and only if X=Y

Lemma 2.2: ([1], Lemma 2.2.3) *Let* $f\in C^{n}([t_0,\infty),\Re^{+})$. Assume that $f^{n}(t)$ is of fixed sign and not identically zero on $[t_0,\infty)$ and that there exists $t_1\geq t_0$ such that $f^{n-1}(t)f^{n}(t)\leq 0$ for all $t\geq t_1$. If $\lim_{t\to\infty} f(t)\neq 0$, then, for every $k\in(0,1)$, there exicts $t_k\in[t_1,\infty)$ such that

$$f(t)\geq\frac{k}{(n-1)!}t^{n-1}\left|f^{n-1}(t)\right|, \text{ for } t\in[t_k,\infty).$$

Lemma 2.3: Let $\alpha\geq 1$, be a ratio of odd positive integers. Then

$$Bu-Au^{\frac{\alpha+1}{\alpha}}\leq\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}, \qquad A,B>0, \tag{12}$$

Now we present the main theorem.

Theorem 2.1: Assume $(H_1)-(H_5)$ and (11) hold. If $\alpha\geq 1$ and there exists a positive no decreasing function $\rho\in C^{'}([t_0,\infty),R)$ such that

$$Lim_{t\to\infty}\sup\int_{t_1}^{t}\left[\rho(s)f(s)\left\{\frac{1}{m(s-\sigma)}\left(1-\sum_{i=1}^{n}\frac{1}{(s-a)(s-\sigma-1)}\left\{1+\frac{1}{c}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right\}\right)\right\}^{\alpha}.\frac{(s-\sigma)^{2\alpha}}{s^{2\alpha}} - \frac{2^{\alpha}}{(\alpha+1)^{\alpha}}.\frac{r(s)(\rho'(s))^{\alpha+1}}{\left(\rho(s)ks^{2}\right)^{\alpha}}\right]ds=\infty \tag{13}$$

for some $k \in (0,1)$, then every solution of equation (1) is oscillatory.

Proof. Suppose to the contrary .And let *y(t)* be a nonoscillatory solution of equation (1).Without loss of generality we may assume that *y(t)* is eventually positive.

Since $z(t) > 0, \quad z'(t) > 0, \quad z''(t) > 0, z'''(t) > 0, z^4(t), (r(t)z'''(t))^\alpha \le 0;$ for $t \ge t_1$

(14)

From (14) and also since $t - \sigma \le t$ we have

$$r(t)z'''(t) \le r(t-\sigma)z'''(t-\sigma) \quad \text{for } t \ge t_1$$

From the definition of z, we have

$$z(t) = m(t)y(t) + \sum_{i=1}^{n} \frac{1}{t(t-1)} y^\alpha (t-\tau)$$

$$m(t)y(t) = z(t) - \sum_{i=1}^{n} \frac{1}{t(t-1)} y^\alpha (t-\tau)$$

$$y(t) = \frac{1}{m(t)} \left[z(t) - \sum_{i=1}^{n} \frac{1}{t(t-1)} y^\alpha (t-\tau) \right]$$

$$y(t) = \frac{1}{m(t)} \left[z(t) - \left\{ \sum_{i=1}^{n} \frac{1}{t(t-1)} y^\alpha (t-\tau) - \sum_{i=1}^{n} \frac{1}{t(t-1)} y(t-\tau) \right\} \right] - \sum_{i=1}^{n} \frac{1}{t(t-1)} y(t-\tau) \tag{15}$$

Also from (15)

$$\sum_{i=1}^{n} \frac{1}{t(t-1)} y^\alpha (t-\tau) - \sum_{i=1}^{n} \frac{1}{t(t-1)} y(t-\tau) \le \sum_{i=1}^{n} (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} . \frac{1}{t(t-1)} \tag{16}$$

Substituting (16) into (15)

$$y(t) \ge \frac{1}{m(t)} \left[z(t) - \sum_{i=1}^{n} \frac{1}{t(t-1)} y(t-\tau) - (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} \sum_{i=1}^{n} \frac{1}{t(t-1)} \right] \tag{17}$$

Since *z(t)>0, z'(t)>0* on $[t_2, \infty)$ then there exists $t_3 \ge t_2$ and a constant c>0 such that

$$y(t) \ge c \;\; for \; t \ge t_3 \tag{18}$$

In view of (18) and the fact that $y(t) \le z(t)$ (17) yields

$$y(t) \geq \frac{1}{m(t)}\left[z(t) - \sum_{i=1}^{n}\frac{1}{t(t-1)}y(t) - (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\sum_{i=1}^{n}\frac{1}{t(t-1)}\right] \text{ since } t-\tau \leq t$$

$$y(t) = \frac{1}{m(t)}\left[z(t) - \sum_{i=1}^{n}\frac{1}{t(t-1)}z(t) - (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\sum_{i=1}^{n}\frac{1}{t(t-1)}\right] \tag{19}$$

$$y(t) = \frac{1}{m(t)}\left[1 - \sum_{i=1}^{n}\frac{1}{t(t-1)} - \frac{1}{z(t)}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\sum_{i=1}^{n}\frac{1}{t(t-1)}\right]z(t) \tag{20}$$

$$y(t) \geq \frac{1}{m(t)}\left[1 - \sum_{i=1}^{n}\frac{1}{t(t-1)} - \frac{1}{c}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\sum_{i=1}^{n}\frac{1}{t(t-1)}\right]z(t)$$

$$y(t) = \frac{1}{m(t)}\left[1 - \sum_{i=1}^{n}\frac{1}{t(t-1)}\left(1 + \frac{1}{c}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right)\right]z(t) \tag{21}$$

From equation (1) we see that

$$\frac{d}{dt}\left\{r(t)\frac{d^3}{dt^3}\left(m(t)y(t) + \sum_{i=1}^{n}\frac{1}{t(t-1)}y^{\alpha}(t-\tau)\right)\right\} = -f(t)y^{\alpha}(t-\sigma)$$

$$y^{\alpha}(t-\sigma) = \left\{\frac{1}{m(t-\sigma)}\left[1 - \sum_{i=1}^{n}\frac{1}{(t-\sigma)(t-\sigma-1)}\left(1 + \frac{1}{c}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right)\right]\right\}^{\alpha} z^{\alpha}(t-\sigma) \tag{22}$$

Define

$$\omega(t) = \rho(t)\frac{r(t)(z'''(t))^{\alpha}}{z^{\alpha}(t)}; \quad t \geq t_1 \tag{23}$$

$$\omega'(t) = \rho'(t)\frac{r(t)(z'''(t))^{\alpha}}{z^{\alpha}(t)} + \rho(t)\left\{\frac{r(t)z'''(t)^{\alpha}}{z^{\alpha}(t)}\right\}'$$

$$\omega'(t) = \rho'(t)\frac{r(t)(z'''(t))^{\alpha}}{z^{\alpha}(t)} + \rho(t)\left[\frac{z^{\alpha}(t)\{r(t)(z'''(t))^{\alpha}\}' - (r(t)z'''(t))^{\alpha}\{z^{\alpha}(t)\}'}{z^{2\alpha}(t)}\right]$$

$$= \rho'(t)\frac{r(t)(z'''(t))^{\alpha}}{z^{\alpha}(t)} + \rho(t)\frac{\{r(t)(z'''(t))^{\alpha}\}'}{z^{\alpha}(t)} - \left[\rho(t)\frac{r(t)(z'''(t))^{\alpha}\{z^{\alpha}(t)\}'}{z^{2\alpha}(t)}\right]$$

From (23) we have,

$$\frac{\omega(t)}{\rho(t)} = \frac{r(t)\left(z'''(t)\right)^{\alpha}}{z^{\alpha}(t)};$$

$$\omega'(t) \le \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)f(t)\left\{\frac{1}{m(t-\sigma)}\left(1-\sum_{i=1}^{n}\frac{1}{(t-\sigma)(t-\sigma-1)}(1+\frac{1}{c}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right)\right\}^{\alpha}\frac{z^{\alpha}(t-\sigma)}{z^{\alpha}(t)} - \left[\rho(t)\frac{r(t)\left(z'''(t)\right)^{\alpha}\left\{z^{\alpha}(t)\right\}'}{z^{2\alpha}(t)}\right]$$

$$\omega'(t) \le \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)f(t)\left\{\frac{1}{m(t-\sigma)}\left(1-\sum_{i=1}^{n}\frac{1}{(t-\sigma)(t-\sigma-1)}(1+\frac{1}{c}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right)\right\}^{\alpha}\frac{z^{\alpha}(t-\sigma)}{z^{\alpha}(t)} - \left[\rho(t)r(t)\left(z'''(t)\right)^{\alpha}.\frac{\alpha z^{\alpha}(t)z'(t)}{z(t).z^{\alpha}(t)}.\frac{1}{z^{\alpha}(t)}\right]$$

$$\le \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)f(t)\left\{\frac{1}{m(t-\sigma)}\left(1-\sum_{i=1}^{n}\frac{1}{(t-\sigma)(t-\sigma-1)}(1+\frac{1}{c}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right)\right\}^{\alpha}\frac{z^{\alpha}(t-\sigma)}{z^{\alpha}(t)} - \left[\alpha\rho(t)r(t)\left(z'''(t)\right)^{\alpha}.\frac{z'(t)}{z(t).z^{\alpha}(t)}.\right]$$

$$\le \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)f(t)\left\{\frac{1}{m(t-\sigma)}\left(1-\sum_{i=1}^{n}\frac{1}{(t-\sigma)(t-\sigma-1)}(1+\frac{1}{c}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right)\right\}^{\alpha}\frac{z^{\alpha}(t-\sigma)}{z^{\alpha}(t)} - \left[\alpha\rho(t)r(t)\left(z'''(t)\right)^{\alpha}.\frac{z'(t)}{.z^{\alpha+1}(t)}.\right]$$

$$z(t) > 0, \quad z'(t) > 0, \quad z''(t) > 0, z'''(t) > 0, z^{4}(t), \left(r(t)z'''(t)\right)^{\alpha} \le 0; \text{ for } t \ge t_1$$

By Kiguradze Lemma [8] we find $z(t) \ge \frac{t}{2}z'(t)$ and hence $\frac{z(t-\sigma)}{z(t)} \ge \frac{(t-\sigma)^2}{t^2}$ (24)

It follows from **Lemma 2.2** that

$$z'(t) \ge \frac{k}{2}t^2 z'''(t) \tag{25}$$

For every $k \in (0,1)$ and all sufficiently large t. Hence by (24) and (25) we have

$$\omega'(t) \le \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t) f(t)\left\{\frac{1}{m(t-\sigma)}\left(1-\sum_{i=1}^{n}\frac{1}{(t-\sigma)(t-\sigma-1)}(1+\frac{1}{c}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right)\right\}^{\alpha}\frac{(t-\sigma)^{2\alpha}}{t^{2\alpha}}$$
$$-\left[\alpha\frac{k}{2}t^{2}z'''(t)\rho(t)\frac{r(t)\left(z'''(t)\right)^{\alpha}}{z^{\alpha+1}(t)}..\right]$$

$$\omega'(t) \le \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t) f(t)\left\{\frac{1}{m(t-\sigma)}\left(1-\sum_{i=1}^{n}\frac{1}{(t-\sigma)(t-\sigma-1)}(1+\frac{1}{c}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right)\right\}^{\alpha}\frac{(t-\sigma)^{2\alpha}}{t^{2\alpha}}$$
$$-\left[\alpha\frac{k}{2}t^{2}z'''(t)\rho(t)\frac{r(t)\left(z'''(t)\right)^{\alpha}}{z^{\alpha}(t).z(t)}..\right]$$

Also from (23) we have,

$$\frac{\left(z'''(t)\right)^{\alpha}}{z^{\alpha}(t)} = \frac{\omega(t)}{\rho(t)r(t)}$$

and
$$\frac{z'''(t)}{z(t)} = \frac{\omega^{\frac{1}{\alpha}}(t)}{\rho^{\frac{1}{\alpha}}(t)r^{\frac{1}{\alpha}}(t)} \Rightarrow z'''(t) = \frac{\omega^{\frac{1}{\alpha}}(t)}{\rho^{\frac{1}{\alpha}}(t)r^{\frac{1}{\alpha}}(t)}z(t)$$

$$\omega'(t) \le \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t) f(t)\left\{\frac{1}{m(t-\sigma)}\left(1-\sum_{i=1}^{n}\frac{1}{(t-\sigma)(t-\sigma-1)}(1+\frac{1}{c}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right)\right\}^{\alpha}\frac{(t-\sigma)^{2\alpha}}{t^{2\alpha}}$$
$$-\frac{\alpha k}{2}t^{2}\frac{\omega^{\frac{1}{\alpha}}(t)}{\rho^{\frac{1}{\alpha}}(t)r^{\frac{1}{\alpha}}(t)}\rho(t)\frac{r(t)\left(z'''(t)\right)^{\alpha}}{z^{\alpha}(t).z(t)}z(t)$$

$$\le \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t) f(t)\left\{\frac{1}{m(t-\sigma)}\left(1-\sum_{i=1}^{n}\frac{1}{(t-\sigma)(t-\sigma-1)}(1+\frac{1}{c}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right)\right\}^{\alpha}\frac{(t-\sigma)^{2\alpha}}{t^{2\alpha}}$$
$$-\frac{\alpha k}{2}t^{2}\frac{\omega^{\frac{1}{\alpha}}(t)}{\rho^{\frac{1}{\alpha}}(t)r^{\frac{1}{\alpha}}(t)}\rho(t)\frac{\omega(t)}{\rho^{\frac{1}{\alpha}}r^{\frac{1}{\alpha}}}\frac{z(t)}{z(t)}$$

$$\le \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t) f(t)\left\{\frac{1}{m(t-\sigma)}\left(1-\sum_{i=1}^{n}\frac{1}{(t-\sigma)(t-\sigma-1)}(1+\frac{1}{c}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right)\right\}^{\alpha}\frac{(t-\sigma)^{2\alpha}}{t^{2\alpha}} - \frac{\alpha k t^2}{2}\frac{\omega^{\frac{1}{\alpha}+1}(t)}{(\rho(t)r(t))\frac{1}{\alpha}} \tag{26}$$

We set $A = \dfrac{\alpha k t^2}{2(r(t)\rho(t))^{\frac{1}{\alpha}}}$, $B = \dfrac{\rho'(t)}{\rho(t)}$,

Using the Inequality

$$Bu - Au^{\frac{\alpha+1}{\alpha}} \le \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}, \qquad A, B > 0,$$

We have

$$\frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{\alpha k t^2}{2(\rho(t)r(t))\frac{1}{\alpha}}\omega^{\frac{\alpha+1}{\alpha}}(t) \le \frac{\left(\frac{\rho'(t)}{\rho(t)}\right)^{\alpha+1}}{(\alpha+1)^{\alpha}.\frac{(\alpha k t^2)^{\alpha}}{2^{\alpha}(r(t)\rho(t))}} \tag{27}$$

$$\le \frac{2^{\alpha}}{(\alpha+1)^{\alpha}}.\frac{r(t)}{\rho^{\alpha}(t)}.\frac{(\rho'(t))^{\alpha+1}}{k^{\alpha}t^{2\alpha}}$$

$$\le \frac{2^{\alpha}}{(\alpha+1)^{\alpha}}.\frac{r(t)(\rho'(t))^{\alpha+1}}{(\rho(t)kt^2)^{\alpha}}$$

We find that

$$\frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{\alpha k t^2}{2(\rho(t)r(t))\frac{1}{\alpha}}\omega^{\frac{\alpha+1}{\alpha}}(t) \le \frac{2^{\alpha}}{(\alpha+1)^{\alpha}}.\frac{r(t)(\rho'(t))^{\alpha+1}}{(\rho(t)kt^2)^{\alpha}} \tag{28}$$

Hence we obtain ,

$$\omega'(t) \le -\rho(t) f(t)\left\{\frac{1}{m(t-\sigma)}\left(1-\sum_{i=1}^{n}\frac{1}{(t-\sigma)(t-\sigma-1)}(1+\frac{1}{c}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right)\right\}^{\alpha}\frac{(t-\sigma)^{2\alpha}}{t^{2\alpha}} + \frac{2^{\alpha}}{(\alpha+1)^{\alpha}}.\frac{r(t)(\rho'(t))^{\alpha+1}}{(\rho(t)kt^{2})^{\alpha}} \quad (29)$$

which implies that on integrating from t_1 to t we get

$$\int_{t_1}^{t}\left[\rho(s) f(s)\left\{\frac{1}{m(s-\sigma)}\left(1-\sum_{i=1}^{n}\frac{1}{(s-\sigma)(s-\sigma-1)}(1+\frac{1}{c}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right)\right\}^{\alpha}\frac{(s-\sigma)^{2\alpha}}{s^{2\alpha}} + \frac{2^{\alpha}}{(\alpha+1)^{\alpha}}.\frac{r(s)(\rho'(s))^{\alpha+1}}{(\rho(s)ks^{2})^{\alpha}}\right]ds \le \omega(t_1)$$

For every $k \in (0,1)$ and sufficiently large t which contradicts to equation (13) as $t \to \infty$. Thus the proof is completed.

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