Ill-Posed Problems

K.Saranya[#1], Ms.N.Rajakumari [*2]

[#1]*M.Phil (Scholar)& Mathematics & PRIST University*
Vallam, Thanjavur, Tamilnadu, India.
[*2]*M.Sc.,M.Phil,Asst.Prof & M.Phil (Mathematics) & PRIST University*
Vallam, Thanjavur, Tamilnadu, India.

Abstract ***—*** *This process proposed to ill-posed problems in the field of mathematical physics and partial differential and integral equations, there are many simpler yet not less important ill-posed problems among algebraic equations, differential equations, extremum problems, etc. To avoid errors, prior to solving any problem it is recommended to check if the problem is a well-posed or ill-posed one. This journal aims to present original articles on the theory, numerics and applications of inverse and ill-posed problems. These inverse and ill-posed problems arise in mathematical physics and mathematical analysis, geophysics, acoustics, electrodynamics, tomography, medicine, ecology, financial mathematics etc.*

Keywords — *ill-posed problems, differential equations, roots of polynomials.*

I. INTRODUCTION

As is well-known, a problem is said to be well-posed in the sense of Hadamard when a unique solution exists and depends continuously upon the data. The def- inition is made precise by stipulating not only the function spaces in which the solution and data are to lie but also the measures and notion of continuity. A problem that is not well-posed is said to be ill-posed. The subject came to prominence only after Hadamard had formulated his well-known definition. His objections were grounded in his celebrated counter example of the Cauchy problem for Laplace's equation. In order for there to be global existence of the solution, Hadamard demonstrated that the Cauchy data must satisfy a certain compatibility relation but even in the unlikely event of the relation being satisfied he further showed that the solution in general does not depend continuously on the data. Such behaviour convinced Hadamard that ill-posed problems lacked physical relevance and hence should be ignored. This became the prevailing attitude, and consequently, in partial diferential equations atleast, activity became confined to the standard initial boundary value prob- lems.Consider the simple Dirichlet problem for a linear elliptic homogeneous dif- ferential equation. Conditions are known guaranteeing that the solution exists, is unique and depends continuously upon the Dirichlet data, i.e., the problem is well-posed. These conditions include the requirement that the solution be spec- ified in a suitable sense at all points of the boundary of the region of definition. Yet, rarely, if ever, this specification can be completely achieved in practice. Mea- suring devices record only approximate values and in any case are able to measure data only at a discrete number of points and not over the entire boundary as demanded by the mathematical theory. The solution is therefore not uniquely determined by the measured values of the data and consequently cannot depend continuously on them. Thus, when subjected to the limitations of the measuring device, even standard problems in diff erential equations are liable to be ill-posed. There are, of course, many other examples of ill-posed problems. In diff eren- tial equations the most frequently cited include not only the Cauchy problem for Laplace's equation but also the backward heat equation, the Dirichlet problem for the wave equation, and the wave and parabolic equations subject to data on time- like regions. It must be emphasised that they all serve as models for practical problems. For instance, the Cauchy problem for Laplace's equation corresponds to the situation, encountered in geophysics, surveying and mineral prospecting, where only part of the boundary is accessible for the measurement of data, but over which an abundance of data can be collected. Further examples arise from inaccuracies in the measurement of the geometry of the region of definition and also the value of the operator. Again, the coefficients themselves in the diff eren- tial equations and boundary operators are part of the data and as such are also subject to measurement errors. Lack of precision in determining the coefficients casts doubt on the validity of supposing that an that a comprehensive theory of ill-posed problem should also include diff erential equation of indefinite type. Ill-posed problems also occur in many other branches of mathematics, ele- mentary examples being the Fredholm integral equation of the first kind, ana- lytic continuation of a function, determination of the derivative of a function that is only approximately specified, and a singular linear system of algebraic equa- tions. A further important class concerns inverse problems where it is typically required to determine the coefficients of an equation from a knowledge of certain functionals of the solution. A well-known example is the one-dimensional inverse Sturm-Liouville problem, in which the value of the ordinary diff erential operator is to be determined from the spectral function of the solution. Other examples arise in inverse scattering theory, while of increasing significance are problems with free boundaries. Tikhonov has shown that a large class of ill-posed problems satisfy a modi- fied definition of well-posedness in which existence and uniqueness are assumed, but in which continuous dependence is required to hold only on some subspace, usually taken to be compact. The latter condition corresponds to the constraint which is imposed when stabilising the problem, while the abstract notion of con- tinuity incorporates the relaxation of the continuity concept.

II. ILL-POSEDNESS

Definition 1.1 A problem is said to be well − posed if the following conditions are fulfilled:

1. The solution exists.
2. The solution is unique.
3. The solution is stable, namely, arbitrarily small variations of coefficients, parameters, initial or boundary conditions give rise to arbitrarily small solution changes. If atleast one of these conditions is not fulfilled then the problem is said to be ill − posed.

Definition 1.2 (Hadamard)
Let U, F be the topological spaces, and A be an operator acting from U in F . Consider an operator equation
$A(u) = f, u \in U, f \in F.$ (1.1)
The problem (1.1) is said to be well − posed in the sense of Hadamard if the following conditions are fulfilled:
For any $f \in F$ there exists an element $u \in U$, such that $A(u) = f$,
i.e., the range R(A) of the operator A coincides with the whole space F .
A solution u of the equation (1.1) is uniquely determined by the element f. In other words, there exists the inverse $A-1$ of the operator A.
The solution u depends continuously on the element F . In otherwords, the operator $A-1$ is continuous.
If atleast one of these conditions is not fulfilled, the problem (1.1) is said to be ill − posed in the sense of Hadamard. Note: Clearly, the problem (1.1) is well-posed in the sence of Hadamard if and only if there exists the continuous inverse $A-1$ of the operator A defined on the whole space F. A typical example of an ill-posed problem is given by the operator equation (1.1) whose operator is linear and compact. In this case, the inverse $A-1$ cannot be defined on the whole space F . Furthermore, it is not continuous even on the set AU . In general, the inverse $A-1$ of the operator A generated by an applied problem cannot be defined on the whole space F . In other words, the third Hadamards condition is extremely strong.
Examples of ill-posed problems
The differentiation problem

Example 1.1 Suppose the function f(x) is given with a noise. In other words, the given function is
$f\delta(x) = f(x) + \delta f(x), \quad x \in [0, 1].$ (1.2)
Then
$\|f\delta - f\|C([0,1]) = \|\delta f\|C([0,1]) \le \delta,$ (1.3)
where δ is small and this number is the level of noise. Then the problem of calculating $f\delta j$ (x) is ill-posed.
For, let us consider $\delta f(x) = \sin nx$.
Now,
Then
$f\delta(x) = f(x) + \sin nx / n \quad , \quad x \in [0, 1].$ (1.4)
$\|f\delta - f\|C([0,1]) = \max |f\delta(x) - f(x)|$
$=\Rightarrow \|f\delta - f\|C([0,1]) = \max | 1$
$=\Rightarrow \|f\delta - f\|C([0,1]) \le n$
and is small for large value of n. sin nx/ n |
Now,
$f\delta j (x) = fj(x) + \cos nx.$
Then
$\|f\delta j - fj\|C([0,1]) = \max |f\delta j (x) - fj(x)|$
$=\Rightarrow \|f\delta j - fj\|C([0,1]) = \max | \cos nx|$
$=\Rightarrow \|f\delta j - fj\|C([0,1]) = 1$ and is certainly not small.
Thus the diff erentiation problem generally does not posses the property of sta- bility in the space C. The third condition of well-posedness is violated according to Hadamard. Hence the diff erentiation problem is an ill-posed problem. By the way, there exists a simple method of regularization of the diff erentiation problem. Indeed,
$f\delta j (x) = fj(x) + \delta fj(x)$
$fj (x) \approx f(x + h) - f(x) + \delta f(x + h) - \delta f(x)$
$=\Rightarrow |fj(x) - f(x + h) - f(x) | \le 1 (|\delta f(x + h)| + |\delta f(x)|)$
$=\Rightarrow |fj(x) - f(x + h) - f(x) | \le 1 (\|\delta f\| + \|\delta f\|)$
$=\Rightarrow |fj (x) - f(x + h) - f(x) | \le 2\delta .$

Hence we take $h = h(\delta) = \delta\mu$, where $\mu \in (0, 1)$, such that $2\delta \lim \delta\to 0\, h(\delta) = 0$.
Hence $h(\delta)$ is the regularization parameter. Basically it says that the mesh step size cannot be too small.

Definition 1.3 (Compact operator)
A linear operator $A : X \to Y$ between normed linear space X and Y is said to be a compact operator if the set $\{Ax : \|x\| \le 1\}$ is compact in Y.

Lemma 1.1 (Lebesque lemma)
If f(t) is a piecewise continuous function on the interval [a, b] then $\int b$
Solution of the integral equation of the first kind
Example 1.2 Let G, $\Omega \subset Rn$ be two bounded domains. Let
$K(x, y) \in C(G \times \Omega)$, $x \in G$, $y \in \Omega$ be a function. Consider the integral operator
$K : C(\Omega) \to C(G)$ defined as
$(Kf)(x) = \Omega$
$K(x, y)f(y)dy, \quad x \in G,$ (1.5)
where K is a compact operator. Consider an integral equation, in which it is not necessary that $\Omega = G$.
$\int K(x, y)f(y)dy = g(x), \quad x \in G.$ (1.6)
This is the so-called the integral equation of the first kind. Show that the problem (1.6) is ill-posed.
For, let $\Omega = (0, 1)$, $G = (a, b)$.
Consider instead of f the function
$fn(x) = f(x) + \sin nx.$ (1.7)
From the equation (1.6) we have,
$\int K(x, y)(fn(y) - \sin ny)dy = g(x)$
Then
$\Longrightarrow \int K(x, y)fn(y)dy = g(x) + K(x, y) \sin nydy.$
$\Omega\ K(x, y)fn(y)dy = gn(x),$ (1.8)
Ω where $gn(x) = g(x) + pn(x)$ and $pn(x) = \Omega\ k(x, y) \sin nydy$.
By Lebesque lemma,
Hence
It is clear that
$\lim \|pn\|C([a,b]) = 0$
$\Longrightarrow \|gn(x) - g(x)\| = 0.$
$\|fn(y) - f(y)\| = 0.$
$\|fn(x) - f(x)\|C([0,1]) = \|\sin nx\|C([0,1])$
$\Longrightarrow \|fn(x) - f(x)\|C([0,1]) \le n\|x\|$ is not small for large n.
The third condition of well-posedness is violated. Hence the problem (1.6) is an ill-posed problem.
Cauchy problem for Laplace equation

Example 1.3 Consider the problem
□ $\Delta u = 0, \quad y > 0,$
□$u(x, 0) = 0,$
□ □$\partial\ u(x, 0) = \alpha \sin nx, \quad x \in [0, \pi].$ (1.9)
Show that the the problem (1.9) is ill-posed.
For, let u(x, y) be the solution of equation (1.9).
Suppose, $u(x, y) = X(x)Y(y)$.
Then
$uxx = XjjY$, $uyy = XY jj$.
So,
$XjjY + XY jj = 0$
Thus,
$Xjj \Longrightarrow - X = Y jj$
$= \lambda$ (say).
Y For $\lambda > 0$,
$X(x) = A \cos \sqrt{\lambda}x + B \sin \sqrt{\lambda}x$, $Y(y) = Ce\sqrt{\lambda}y + De-\sqrt{\lambda}y$, (1.10)
since from $Y(0) = 0$ we have $C + D = 0$. From $Yy(0) = \alpha \sin nx$ we have,
$\sqrt{\lambda}C - \sqrt{\lambda}D = \alpha \sin nx$
$-2\sqrt{\lambda}D = \alpha \sin nx\ \alpha$
$D = - 2\sqrt{\lambda} \sin nx,$
so we obtain,

Thus α
$C = 2\sqrt{\lambda} \sin nx$.
$\alpha\sqrt{\lambda}y - \sqrt{\lambda}y$
$Y(y) = 2\sqrt{\lambda} \sin nx$
$=\Rightarrow Y(y) = \sqrt{\lambda} \sin(nx) \sin h(\lambda y)$.
Also, since from $X(x) = 0$ we have,
$A \cos \sqrt{\lambda}x + B \sin \sqrt{\lambda}x = 0$. (1.11)
From $Xy(x) = \alpha \sin nx$ we have,
$A \cos \sqrt{\lambda}x + B \sin \sqrt{\lambda}x = \alpha \sin nx$. (1.12) From the last two equations (1.11) and (1.12) for $x \in [0, \pi]$ we have,
$x = 0 =\Rightarrow A = 0$.
$x = \pi =\Rightarrow B \sin \sqrt{\lambda}\pi = 0$.
If B = 0 then the solution is trivial.
So,
$\sin \sqrt{\lambda}\pi = 0$
$\sqrt{\lambda}\pi = n\pi$
$\sqrt{\lambda} = n$ $\lambda = n2$.
Hence the solution of the problem (1.9) is
$u(x, y) = \alpha \sin nx \sin h(ny)$. (1.13)
n For any pair of functional spaces Ck, Lp, Hl , W l and any $s > 0$, $c > 0$, $g > 0$
it is possible to choose α and n such that
$\|\alpha \sin nx\| < s$,
But since,
$A \| n \sin nx \sin h(ny)\| > s$,
A $\lim n\rightarrow 0$
$n \sin nx \sin h(ny)\| = \infty$,
i.e., small variations of boundary conditions will result in arbitrarily large vari- ation of u(x, y). Hence this problem does not depend continuously on its data. The third condition of well-posedness according to Hadamard is violated. Hence the problem (1.9) is an ill-posed problem.

III. SIMPLEST ILL-POSED PROBLEMS

System of algebraic equations

The necessity in studying ill-posed problems stems from one of the main problems in applied mathematics, gaining reliable computing results with due allowance for errors that inevitably occur in setting coefficients and parameters of a mathematical model used to perform computations. Indeed, coefficients in a mathematical model, equations, or a set of equations used to perform computations are obtained from measurements; for this reason, they are accurate only to some limited accuracy. Moreover, parameters of an actual process or a technical object under sim- ulation are never perfectly time independent quantities. Instead they undergo uncontrollable changes, or display variations, whose exact value is usually un- known. So we will diff erentiate between the nominal values of coefficients ain and their real, “true” values ait. The nominal values are the values to be fed into the computer and used in all computations, whereas the real, “true” values ait are never known. These unknown values are confined between certain limits and therefore obey some inequalities
$(1 - s)ain \leq ait \leq ain(1 + s)$. (2.1)
Here s is a number small compared to unity.
Thus, the exact values of coefficients are never known and only estimates for the coefficients are available. The products ±s.ain is known as nominal-coefficient variations, or errors in nominal coefficients. Since all computations are always conducted with nominal coefficients, it is necessary to check how coefficient variation aff ect the computing accuracy. There exist problems in which solution error are the same order of magnitude as error in setting coefficients; this case is simplest to treat. However, there are problems wherein solution errors are greater than coefficient errors. Finally, there are problems where even very small, practically unavoidable errors in setting co- efficients, parameters, or initial and boundary conditions give rise to appreciable solution errors. Such problems are called ill-posed ones. Nevertheless, such prob- lems are often encountered in practice, and methods enabling their adequate solution need to be put to scrutiny.
Consider the following over determined system of linear algebraic equation
2y1 3y2 =4,
y1 + 2y2 = 3,
y1 + 4y2 = 15. (2.2)
This system of linear algebraic equation is indeed an over determined system having no solution because the rank of the extended matrix here is
$\rho = rank([A|f]) = 3$ and the rank of A is $r = rank(A) = 2$, i.e., $\rho > r$. In addition, the number of independent rows in (2.2) is 3 which is greater than the number of unknowns n = 2.

The fact that this system of linear algebraic equation has no solutions y1, y2 can be proved immediately. Indeed, on solving the first two equations in (2.2) we obtain the solution

y1 = 1, y2 = 2.
On solving the second and third equations in (2.2) we obtain the solution
y1 = y2 = 3.
If alternatively, on solving the first and third equation in (2.2) we obtain the solution
y1 = 2.635, y2 = 3.09.
There is no solution common to all equations. Here the second condition for well-posedness is violated according to Hadamard.
Example 2.2 Consider the following underdetermined system of linear algebraic equation
$2y1 - 3y2 = -4.$ (2.3)
For this system of linear algebraic equation, the rank of the extended matrix is
$\rho = \text{rank}([A|f]) = 1$ and the rank of A is $r = \text{rank}(A) = 1$, i.e., $\rho = r = 1$.
In addition, the number of independent rows in (2.3) is 1 which is less than the number of unknowns n = 2. Therefore this system of linear algebraic equation is an underdetermined system that has many solutions.
For instance, from equation (2.3), if y1 = 1 then we have y2 = 2, if y1 = 2 then we have y2 = 8 ,
if y1 = 0 then we have y2 = 4 , etc, are the solutions of the system.
Thus, the solution of the system of linear algebraic equation is not-unique. The second condition for well-posedness is violated according to Hadamard.

Definition 3.1 (Ill-conditioned systems)
Systems whose relative solution errors are much greater than relative coefficient errors. Such system is often called ill-conditioned systems.
Example 2.3 Consider a system of equation
$1.1x + y = 1.1,$
$(1 + s)x + y = 1.$ (2.4)
We obtain,
$y = \quad -11s.\ 1 - 10s$
if $|s| \le 0.001$ then $0.99 \le x \le 1.01$, if $|s| \le 0.01$ then $0.909 \le x \le 1.11$, if $|s| \le 0.1$ then $0.5 \le x \le \infty$.
Here the solution error is greater than the coefficient error. Moreover the solution error rapidly grows in value with increasing s and may be arbitrarily large if $|s| \le 0.1$.
Note: Consider the limiting case where, at nominal values of coefficients, the determinant a11 a12 becomes zero. Here, even small coefficient variations may give rise to large, or even dramatically large, changes in the solution.

Example 2.4 Consider the system
$x + y = b1,$
$x + y = 1.$ (2.5)
For this system, we have
If b1 f= 1, then the system has no solutions.
If b1 = 1, then there are many solutions. In this case, the straight lines are coincident, and any pair of numbers x = 1 − y represents a solution.
Now consider the system,
Here,
$(1 + s)x + y = b1,$
$x + y = 1.$ (2.6)
and
$x = b1 - 1 , s$
$y = 1 + s - b1$
s
$\Longrightarrow y = 1 - b1 - 1 .$
If b1 f= 1, then for any s the solution does exists, although it entirely depends on the unknown error s. From the practical point of view, this solution, although existing for any s f= 0, is meaningless.
If b1 = 1, then x = 0, y = 1 also represents a solution. This solution is valid for all values of s (including arbitrarily small s)except for the single value s = 0, to which an infinite set of solutions corresponds. Whether the solution is unique or there are many solutions depends on the unknown value of s.

Problems on finding roots of polynomials
Another example of ill-posed problems is given by problems on finding roots of polynomials in those cases

where these polynomials have multiple roots, but only real solutions are physically meaningful. Consider a simplest second-degree polynomial $x^2 + 2x + 1$. The roots are given
by
$x_{1,2} = -b \pm \sqrt{b^2} - 4ac\ 2a$
$x_{1,2} = -2 \pm \sqrt{4} - 4$
$2x_{1,2} = -1$.
This polynomial has the double root $x_1 = x_2 = -1$.
Yet, if the coefficient at the last term is not unity exactly, i.e., if this coefficient equals 1 + s which always may be the case because all coefficients are known to some limited accuracy, then the real solution for arbitrarily small s > 0 vanishes at once, and we have
$x_{1,2} = -2 \pm 4 - 4(1 + s)\ 2$
$x_{1,2} = -1 \pm \sqrt{-s}$.
If our concern is only with real solutions, then already for arbitrarily small s > 0 the solution vanishes. Hence the problem on finding real valued multiple roots is therefore an ill-posed problem. Boundary value problem for ordinary differential equations Many boundary value problems for ordinary differential equations also possess the property of being ill-posed.
Consider the equation
$d^2x\ dt^2 + x = 0$ (2.7) with the boundary conditions $x(t = 0) = 0$, $x(t = a) = b$. The general solution of (2.7) is
$x = c_1 \sin t + c_2 \cos t$, (2.8) where c_1 and c_2 are constants of integration. Now,
$x(0) = 0 \implies c_2 = 0$
$b\ x(a) = b \implies c_1 = \sin a$.
Thus, the set of boundary conditions are satisfied with the solution
$x_1(t) = b \sin t$. (2.9)
sin a Yet, if the boundary condition is set at the point t = a + s instead of the point t = a, then the solution becomes $b \sin t\ x_2(t) = \sin(a + s)$. (2.10)
If, for instance, $a = \pi - s$, then the absolute difference between $x_1(t)$ and $x_2(t)$ may be arbitrarily large even for arbitrarily small s.
Many important physical and technical problems require calculating the magnitude of some parameter (λ) for which a system of linear homogeneous equations with parameter has non-zero solutions.
Consider the system
$(\lambda^2 - 2\lambda)x_1 + (1 - 3\lambda)x_2 = 0$,
$\lambda x_1 - 3x_2 = 0$. (2.11)
Now finding x_1 and x_2,
$\lambda x_1 - 3x_2 = 0 \implies x_2 = \lambda x_1$.
$(\lambda^2 - 2\lambda)x_1 + (1 - 3\lambda)x_2 = 0$
$\lambda^2 x_1 - 2\lambda x_1 + \lambda x_1$ $\lambda^2 x = 0\ 3$
$5x_1 = 0$
so $x_2 = 0. \implies x_1 = 0$,
Since the system (2.11) is homogeneous, this system has the trivial solution
$x_1 = x_2 = 0$.
Yet, for some values of λ, the system may also have non-zero solutions. For instance, system (2.11) has non-zero solutions if λ = 0. Then, this system assumes the form
$0x_1 + x_2 = 0$,
$0x_1 - 3x_2 = 0$, (2.12) and any pair of numbers in which $x_2 = 0$ and x_1 is an arbitrary number will satisfying the equations (2.11) with λ = 0 identities. The values of λ for which a system of linear homogeneous equations involving a parameter has non-zero solutions are called the eigenvalues. Finding eigenval- ues is an important step in solving systems of linear differential equations with constant coefficients. Consider the following system of equations for the variables y_1 and y_2:
$\ddot{y}_1 - 2\dot{y}_1 = 3\dot{y}_2 - y_2$,
$\dot{y}_1 = 3y_2$. (2.13) Assume that the solutions of (2.13) are functions
$y_1 = x_1 e^{\lambda t}$, $y_2 = x_2 e^{\lambda t}$,
Where, x_1 and x_2 are the constants of integration. From (2.14) we have,
$\dot{y}_1 = \lambda x_1 e^{\lambda t}$, $\ddot{y}_1 = \lambda^2 x_1 e^{\lambda t}$, $\dot{y}_2 = \lambda x_2 e^{\lambda t}$.
Substituting these values in (2.13), we have $\lambda^2 x_1 e^{\lambda t} - 2\lambda x_1 e^{\lambda t} = 3\lambda x_2 e^{\lambda t} - x_2 e^{\lambda t} \implies [(\lambda^2 - 2\lambda)x_1 + (1 - 3\lambda)x_2]e^{\lambda t} = 0$, and $\lambda x_1 e^{\lambda t} = 3x_2 e^{\lambda t} \implies [\lambda x_1 - 3x_2]e^{\lambda t} = 0$.
Having cancelled out the non-zero function $e^{\lambda t}$, we have the system (2.11).
So the non-zero solutions of (2.14) exist for those values of λ for which the system (2.11) has non-zero solutions, i.e., eigenvalues. It is these, and only these, values of λ that can be the exponents in the solutions of the system of linear differential equations with constant coefficients. So finding eigenvalues of a ho- mogeneous linear system of algebraic equations is a necessary stage in solving systems of linear differential equations with

constant coefficients. In turn, the eigenvalues are the roots of the polynomial matrix determinant.
The polynomial matrix for system (2.11) is
$\lambda 2 - 2\lambda \quad 1 - 3\lambda$ (2.15)
The determinant of (2.15) is $\Delta = \lambda \quad -3$ □ $.\lambda 2 - 2\lambda \ 1 - 3\lambda.\ \lambda \quad -3$.
$\Delta = -3\lambda 2 + 6\lambda - \lambda + 3\lambda 2$
$\Delta = 5\lambda.$ (2.16)
This polynomial (2.16) has a single root which is zero. Generally, for a system of differential equations
$A1(D)x1 + A2(D)x2 \quad = 0,$ (2.17)
where D = d
$A3(D)x1 + A4(D)x2 \quad = 0,$
is the differentiation operator, and A1(D), A2(D), A3(D) and A4(D) are polynomials of some degrees, whose eigenvalues are the roots of the determinant
$A1(\lambda)\ A2(\lambda)$
$\Delta = A\ (\lambda)\ A\ (\lambda)$ (2.18)
The eigenvalues are the roots of the polynomial determinant (i.e., of a determinant whose elements are polynomials). For instance, the eigenvalues for the system
$An1(D)x1 + \bullet\bullet\bullet + Ann(D)xn \quad = 0$ are the roots of the determinant
$An1(\lambda)\ \ldots \quad Ann(\lambda)$
The degrees of the operator polynomials in (2.19) depend on the problem under consideration. For instance, in the commonly encountered problem on finding frequencies of small -amplitude oscillations of mechanical or electrical systems, the equations for the oscillations are constructed based on second-kind Lagrange equations. Hence the polynomials $Aij(\lambda)$ are quadratic polynomials. The problems on finding the roots of polynomial determinants similar to (2.20) can be either well- or ill-posed problem. For instance, calculation of (2.16) is an ill-posed problem. Indeed, consider variation of just one coefficient and calculate, instead of (2.16), the determinant
$\Delta = .\lambda 2 - 2\lambda \quad 1 - 3\lambda. \quad \lambda{-}3(1 + s).$
$\Delta = -3\lambda 2 - 3s\lambda 2 + 6\lambda + 6s\lambda - \lambda + 3\lambda 2$
To find the roots, so $\lambda 1 = 0$, and
$\Delta = -3s\lambda 2 + 6s\lambda + 5\lambda.$ (2.21)
$-3s\lambda + 6s\lambda + 5\lambda = 0$
$\Longrightarrow [-3s\lambda + 6s + 5]\lambda = 0,$
$3s\lambda = 6s + 5$
$\Longrightarrow \lambda 2 = 2 + 3s$.
For an arbitrarily small s, determinant (2.21) has two roots, $\lambda 1 = 0$ and $\lambda 2 = 2 + 3s$. As $s \to 0$, the second root by no means tends to the first one and vanishes if s = 0 exactly. For the polynomial-matrix determinants, the reason for the property of being ill-posed is quite clear: this property arises wherever, with nominal values of coefficients, the terms with the highest degree of λ cancel. It is clear that, even with arbitrarily small values of coefficients, we have no such cancellation; as a result, under small parameter variations, the degree of the polynomial of λ in the determinant undergoes changes; as a result, another polynomial root emerges.
Nevertheless, the fact that calculation of determinants of some polynomial matrices presents an ill-posed problem means that some of even more important and commonly encountered problems, problems on solving systems of ordinary differential equations, are also ill-posed problems. The property of being ill-posed may emerge even in solving the simplest class of differential equations often met in applications, namely, linear equations with constant coefficients.
$\ddot{y}1 \quad = 2\dot{y}1 + 3\dot{y}2 - y2,\ \dot{y}1 \quad = my2,$
and analyze how the solutions depend on the parameter m. The characteristic polynomial of system (2.23) is given by the determinant
$\Longrightarrow \Delta = (3 - m)\lambda 2 + (2m - 1)\lambda.$ (2.24)
Suppose that $m \ f{=}\ 3$, then$[(3 - m)\lambda + (2m - 1)]\lambda = 0,$
so $\lambda 1 = 0$, and $(3 - m)\lambda + (2m - 1) = 0$
$\lambda = 2m - 1 - m$
Thus the polynomial has two roots $\lambda = 0$ and $\lambda = 2m{-}1{-}m$
if $m\ f{=}\ 3$.
Suppose that m = 3, then
$5\lambda = 0$
$\lambda 1 = 0.$
Thus the polynomial has only one root $\lambda 1 = 0$ if m = 3. Evidently, the value of m = 3 is singular.Suppose that m = 3(1+s), we are going to analyze now how the solutions depends on s.
Then
so $\lambda 1 = 0$, and $(3 - 3(1 + s))\lambda 2 + (2(3[1 + s]) - 1)\lambda = 0 \Longrightarrow [-3s\lambda + (5 + 6s)]\lambda = 0,\ -3s\lambda = 5 + 6s \Longrightarrow \lambda 2 = 2 + 3s$

The general solution of (2.23) is

$y1(t) = c1e\lambda1t + c2e\lambda2t$ y (t) = c + c e(2+)t. (2.25

The solution y2(t) is of the same form. The constants of integration in (2.25) can be found from boundary conditions. Let the initial conditions be such that c1 = 0 and c2 = 1. Then the magnitude of y1(t), (say) at the time t = 1 as a function of s. If s is a small negative number, then the exponent 2 + 5 is a large (in absolute value) negative number. It follows from here that the solution y1(t) will be close to zero as $t \to \infty$.

If, alternatively, s is a small positive number, then the exponent 2 + 5 is large and the solution y1(t) at t = 1 will be a very large number; this number will be the larger the smaller is s. For $s \to 0$ (i.e., for $m \to 3$), we have y1(t) $\to \infty$. Thus, at the point m = 3 the solution y1(t) as a function of m suff ers discon- tinuity, continuity being voilated. Next, consider practical consequences of this discontinuity. Let the nominal value of m in (2.23) be 2.999, and the error in setting this parameter, be equal to two tenths. Calculating the magnitude of y1(t) at t = 1 (or at any other time t) with the nominal value of m, m = 2.999, we readily obtain that y1(t) = 0 to the fourth decimal point. Yet, with regard for the accuracy in setting m in (2.23), the true value of m can be 3.001, then the true of y1(t) will be large and equally large will be the calculation error, pregnant with bad consequences, wrecks and even catastrophes. The latter is caused by the fact that the problem of finding the solution of (2.23) with m = 3 is an ill-posed problem. Crude errors in calculations will also arise in solving all system of equations whose characteristic polynomial changes its degree at some critical value of an involved parameter, resulting in cancellation of coefficients at the higher degree. If the magnitude of the parameter is not equal to the critical value exactly, but is close to it, then there will be no change of degree in the characteristic poly- nomial, but the coefficient at the highest degree will be small, i.e., the polynomial will acquire the form

$\Delta = s\lambda n + an-1\lambda n-1 + \bullet\bullet\bullet + a0$, (2.26)

where s is a number much smaller than the coefficients an−1, an−2, . . . , a0. Here, by dividing polynomial (2.26) by the binomial sλ an−1 + 1, (2.27)

we can prove that, polynomial (2.26) will have one large (in absolute value) root λn ≈ − an−1 , while the other n − 1 roots will be close to the roots of (2.26) with omitted first term. In residue, we obtain a polynomial close (for small values of s) to polynomial (2.26) with omitted first term. The sign of the larger root λn ≈ − an−1 depends on the sign of s; the root therefore changes its sign to the opposite when s passes through zero. This means that in the case of s < 0 a rapidly growing term will arise in the solution, while with s = 0 the solution as a function of s suffers discontinuity.

Ill-posed problems regarding wronskian

The general solution of an n-th order homogeneous linear equation y(n) + pn−1(x)y(n−1) + • • • + p0(x)y = (3.1) has the particularly simple form n y(x) = cjyj(x), (3.2) j=1 where the cj are arbitrary constants of integration and {yj(x)} is a linearly inde- pendent set of functions, and each satisfying (3.1). There are always exactly n linearly independent solutions to (3.1) in any region where the coefficient func- tions p0(x), p1(x), . . . , pn−1(x) are continuous. The Wronskian W (x) is defined as the determinant W (x) = W [y1(x), y2(x), . . . , yn(x)]

Homogeneous linear equations have a remarkable property: the Wronskian W (x) of any n solutions of (3.1) satisfies the simple first-order equation W j(x) = −pn−1(x)W (x). (3.4) The solution of (3.4) is known as Abel's formula: x W (x) = exp − pn−1(t)dt .(3.5) Thus, we have the surprising result that W (x) can be computed before any of the solutions of the diff erential equation are known. The indefinite integral in Abel's formula means that W (x) is determined up to an arbitrary multiplicative constant. Choosing a new set of n solutions (which, of course, will all be linear combinations of the old set) merely alters the constant.Let us now use these theoretical results to discuss the well-posedness of initial- value and boundary-value problems. Well-posedness is a concept which is usually associated with partial diff erential equations, but it is also appropriate here.

Initial-value problems

To solve an initial-value problem one must choose the cj in (3.2) so that the initial conditions in (3.1) are satisfied. The cj are determined by a set of n simultaneous algebraic equations

n(i) j=1 But, according to Cramer's rule, these equations have a unique solution only if det[y(i)(x0)] = W (x0) *f*= 0.(3.6) Thus, the Wronskian appears naturally in the study of initial-value problems. It actually has two related but distinct diagnostic applications. First, it may be used globally to determine whether a solution of the form in (3.2) is in fact the general solution of (3.1) by testing whether {yj} is a linearly independent set. In fact, the exponential form of the Wronskian implies that the general solution in one region remains the general solution in any region which can be replaced without passing through singularities of the coefficient functions, i.e., the exponential in (3.5) can never vanish except possibly at a singularity of pn−1(x). Second, the Wronskian may be used locally to spot an ill-posed initial-value problem without actually solving the diff erential equation by simply evaluating (3.5) and referring to (3.6). A homogeneous initial-value problem is ill-posed if the initial conditions are given at a point x0 for which the Wronskian, as calculated by Abel's formula, vanishes; either there is no solution at all or else there are infinitely many solutions.

IV. CONCLUSION

Apart from the well-known ill-posed problems in the field of mathematical physics and partial differential and integral equations, there are many simpler yet not less important ill-posed problems among algebraic equations, differential equations, extremum problems, etc. To avoid errors, prior to solving any problem it is recommended to check if the problem is a well-posed or ill-posed one.

REFERENCES

[1] Carl M. Bender, Steven A. Orszag, Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory, Springer, 1991.
[2] Heinz W. Engl, Martin Hanke and Andreas Neubauer, Regulariza- tion of Inverse Problems, Kluwer Academic Publishers, 1996.
[3] V. K. Ivanov, V. V. Vasin, V. P. Tanana, Theory of Linear Ill-posed Problems and its Applications, Walter De Gruyter GmbH and Co, 2002.
[4] A. Kirsch, An Introduction to the Mathematical Theory of Inverse Prob- lems, Springer-Verlag, New York, Inc, 1996.
[5] Peter Linz, A New Numerical Method for Ill-posed Problems, Inverse Prob- lems 10, L1-L6, 1994.
[6] Yu. P. Petrov, V. S. Sizikov, Well-posed, Ill-posed, and Intermediate Problems with Applications, Brill Academic Publishers, 2005.
[7] A. A. Samarskii, P. N. Vabishchevich, Numerical Methods for Solving Inverse Problems of Mathematical Physics, Walter De Gruyter GmbH and Co, 2000.