Results On A-Unitary, A-Normal and A-Hyponormal Operators

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Abstract: In this paper, properties of the automorphism class G_A of A −unitary, A −normal and A −hypernormal operators on a Hilbert space are investigated. In this context, A is a self-adjoint and an invertible operator. It is also proved that A −unitary equivalence is an equivalence relation. More results on A −unitary operators are also proved in terms of the polar decomposition of an operator T. Finally, A-hyponormal operators are stated and then prove the result that an A-skew- adjoint opetator is A −unitary but not unitary.

Keywords and Phrases: A-self- adjoint, A-unitary, Hilbert space, A − unitary equivalence, A-skew- adjoint operators.

I. INTRODUCTION

In this research thesis Hilbert spaces or subspaces will be denoted by capital letters, H, H_1, H_2, K, K_1, K_2 etc and T, T_1, T_2, A, B, denote bounded linear operators where an operator means a bounded linear transformation. $B(H)$ will denote the bounded linear operators on a complex separable Hilbert space H. $B(H,K)$ denotes the set of bounded linear transformations from H to K, which is equipped with the (induced uniform) norm.The following definitions are of essence:

Definition 1.1: Let H be a linear (vector) space over a field $K \in \{\mathbb{R}, \mathbb{C}\}$.
An *inner product* is a bilinear function $\langle \, , \, \rangle : H \times H \longrightarrow \mathbb{C}$ with the following properties:

1. $\langle ax + by, z \rangle = a\langle x, z\rangle + b\langle y, z\rangle \ \forall \ x, y, z \in H$ and $a, b \in K$, that is, linearity to the first argument is satisfied;
2. $\langle z, ax + by\rangle = \bar{a}\langle z, x \rangle + \bar{b}\langle z, y \rangle \ \forall \ x, y, z \in H$ and $a, b \in K$, that is, semi-linearity to the second argument is satisfied;
3. $\langle x, y \rangle = \overline{\langle y, x \rangle} \ \forall \ x, y \in H$. This property is called the complex conjugation;
4. $\langle x, x \rangle \geq 0 \ \forall \ x \in H$ and $\langle x, x \rangle = 0$ if and only if $x = 0$. This is the non-negative (or positive definite) property.

A linear space equipped with an inner product is called an inner product space. This will be denoted by the set $(H, \langle ., . \rangle)$. A *Hilbert space* is a complete inner product space. The norm $\| x \|$ of a vector $x \in H$ is defined as the positive square-root $\| x \| = \langle x, x \rangle^{1/2}$.
We note that the restriction of the bilinear function $\langle ., . \rangle$ to a subspace $K \subset H$ satisfies the properties of an inner product and by this fact, every subspace of an inner product space is itself an inner product space.

Definition 1.2: If $T \in B(H)$ then its adjoint T^* is the unique operator in $B(H)$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ $\forall \ x, y \in H$.

Definition 1.3: A contraction on H is an operator $T \in B(H)$ such that $T^*T \leq I$ (i.e. $\| Tx \| \leq \| x \| \ \forall \ x \in H$). A *strict or proper* contraction is an operator T with $T^*T < I$ (i.e. $\sup_{0 \neq x} \frac{\|Tx\|}{\|x\|} < 1$). If $T^*T = I$, then T is called *a non-strict contraction.*

Definition 1.4: An operator $T \in B(H)$ which is self adjoint is said to be positive if $\langle Tx, x \rangle \geq 0 \ \forall \ x \in H$.

Definition 1.5: An operator $T \in B(H)$ is said to be isometric if $T^*T = I$.

Definition 1.6: An operator $T \in B(H_1,H_2)$ is said to be invertible if there exists an operator $T^{-1} \in B(H_2,H_1)$ such that $T^{-1}Tx = x$ for every $x \in H_1$ and $TT^{-1}y = y$ for every $y \in H_2$. The operator T^{-1} is called the inverse of T.

Definition 1.7: Suppose $A \in B(H)$ is a positive operator, then an operator $T \in B(H)$ is called an $A-contraction$ on H if $T^*AT \leq A$. If equality holds, that is $T^*AT = A$, then T is called an $A-isometry$. Here A is a self adjoint and invertible operator. Such operators were extensively studied by Suciu [19].

Definition 1.8: Let T be a linear operator on a Hilbert space H. We define the $A-adjoint$ of T to be an operator S such that $AS = T^*A$ where A is self adjoint and invertible.

Remark 1.9: The existence of such an operator in the above definition is not guaranteed. It may or may not exist. In fact a given $T \in B(H)$ may admit many A –adjoints and if such an $A-adjoint$ of T exists, we denote it as $T^{[*]}$. Thus $AT^{[*]} = T^*A$. Since A is invertible $T^{[*]} = A^{-1}T^*A$.
We also need the following terminologies in this paper:
An operator $T \in B(H)$ is said to be:
an $involution$ if $T^2 = I$,
$self - adjoint$ or $Hermitian$ if $T^* = T$ or equivalently, if $\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \forall\ x,y \in H$,
$unitary$ if $T^*T = TT^* = I$,
a $projection$ if $T^2 = T$ and $T^* = T$,
$isometric$ if $T^*T = I$,
a $symmetry$ if $T^* = T = T^{-1}$, that is T is a self-adjoint unitary,
$normal$ if $T^*T = TT^*$ (equivalently, if $\| Tx \| = \| T^*x \| \quad \forall\ x \in H$),
If H and K are Hilbert spaces, then their (orthogonal) direct sum will be denoted by $H \oplus K$, which itself is a Hilbert space.
By a subspace of a Hilbert space H we mean a closed linear manifold of H, which is also a Hilbert space. If M and N are orthogonal (denoted by $M \perp N$) subspaces of a Hilbert space H, then their (orthogonal) direct sum $M \oplus N$ is a given subspace of H. For any set $M \subseteq H$, $M^\perp$ will denote the orthogonal complement of M in H which is a subspace of H. If M is a subspace of H, then H can be decomposed as $H=M\oplus M^\perp$.
A set M in H is invariant for T if $T(M) \subseteq M$. M is an invariant subspace for T if it is a subspace of H which, as a subset of H, is invariant for T. A subspace M of H is invariant for T if and only if $M^\perp$ is invariant for T.
A subspace M reduces T (or M is a reducing subspace for T) if both M and $M^\perp$ are invariant under T(equivalently, if M is invariant for both T and T^*).
If M is an invariant subspace for T then, relative to the decomposition $H = M\oplus M^\perp$, the operator T can be written as

$$T=\begin{bmatrix} T|_M & X \\ 0 & Y \end{bmatrix} \quad \text{for operators } X\text{: } M^\perp \longrightarrow M \text{ and } Y : M^\perp \longrightarrow M^\perp,$$

where $T|_M : M \longrightarrow M$ is the restriction of T on M.
A $part$ of an operator T is a restriction of T to an invariant subspace. Conversely, if an operator T on H can be written as the triangulation $T=\begin{bmatrix} Z & X \\ 0 & Y \end{bmatrix}$ in terms of the decomposition $H=M\oplus M^\perp$, then $Z = T|_M : M \to M$ is a part of T. $X = 0$ if and only if M reduces T. In a such a case, the operator T is decomposed (reduced) into the (orthogonal) direct sum of the operators $Z = T|_M$ and $Y = T|_{M^\perp}$:$T = Z \oplus Y$. With respect to the decomposition $H=M\oplus M^\perp$, the projection onto M (i.e. the unique projection $P : H \longrightarrow H$ such that $Ran(P) = M$ can be written as $P=\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. Therefore, M is invariant if and only if $PTP = TP$ and M reduces T if and only if $PT = TP$.
An operator $T \in B(H)$ is said to be $subnormal$ if it has a normal extension. That is, if there exists a normal operator N on a Hilbert space K such that H is a subspace of K and the subspace H is invariant under the operator N and the restriction of N to H coincide with T. That is
$T = N|_H$, i.e $N = \begin{bmatrix} T & X \\ 0 & Y \end{bmatrix}$ is normal , where $X \in B(H^\perp, H)$, $Y \in B(H^\perp)$ and $K = H \oplus H^\perp$.
Let H be a Hilbert space and $T \in B(H)$. The set $\rho(T)$ of all complex number λ for which $(\lambda I - T)$ is invertible is called the $resolvent\ set\ of\ T$. Equivalently,

$$\rho(T) = \{ \lambda \in \mathbb{C} : Ker\ (\lambda I - T) = \{0\} \text{ and } Ran\ (\lambda I - T) = H \}.$$

The complement of the resolvent set $\rho(T)$ denoted by $\sigma(T)$ is called the $spectrum\ of\ T$ i.e,

$\sigma(T) = C\backslash\rho(T) = \{\lambda \in \mathbb{C}: Ker\,(\lambda I - T) \neq \{0\}\ or\ Ran(\lambda I - T) \neq H\}$, which is the set of all λ such that $(\lambda I - T)$ fails to be invertible (i.e. fails to have a bounded inverse on $Ran(\lambda I - T) = H$). On the basis of this failure, the spectrum can be split into many disjoint parts. A classical disjoint partition comprises of three parts: the set of those $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ has no inverse, denoted by $\sigma_P(T)$ is called the point spectrum of, T i.e,

$\sigma_p(T) = \{\lambda \in \mathbb{C}: Ker\,(\lambda I - T) \neq \{0\}\}$, which is exactly the set of all eigenvalues of T.

The set of all those $\lambda \in \mathbb{C}$ for which $(\lambda I - T)$ has a densely defined but unbounded inverse on its range, denoted by $\sigma_c(T)$ is called the continuous spectrum of T, i.e,

$\sigma_c(T) = \{\lambda \in \mathbb{C}: Ker\,(\lambda I - T) = \{0\},\ \overline{Ran(\lambda I - T)} = H$ and $Ran(\lambda I - T) \neq H\}$.

If $(\lambda I - T)$ has an inverse that is not densely defined, then λ belongs to the residual spectrum of T, denoted $\sigma_r(T)$. That is, $\sigma_r(T) = \{\lambda \in \mathbb{C}: Ker\,(\lambda I - T) = \{0\},\ \overline{Ran(\lambda I - T)} \neq H\}$.

The parts $\sigma_p(T)$, $\sigma_c(T), and\ \sigma_r(T)$ are pairwise disjoint and $\sigma(T) = \sigma_p(T) \cup \sigma_r(T) \cup \sigma_c(T)$.

An operator $T \in B(H)$ is said to be $hyponormal$ if $T^*T \geq TT^*$ or equivalently if $T^*T - TT^* \geq 0$ (i.e., the self-commutator of T is a non-negative operator).

Two operators $T \in B(H)$ and $S \in B(K)$ are *unitarily equivalent* (denoted $T \cong S$), if there exists a $unitary$ operator $U \in \mathcal{G}\,(H,K)$ such that $UT = SU$ $(i.e, T = U^*SU\ or\ equivalently\ S = UTU^*)$.

Suppose $A \in B(H)$ is a $self-adjoint$ and $invertible$ operator, not necessarily unique.

An operator $T \in B(H)$ is said to be:

$A-self-adjoint\ if\ T^* = ATA^{-1}$ (equivalently, $T^{[*]} = T$).

$A-skew-adjoint\ if\ T^* = -ATA^{-1}$ (Equivalently, $T^{[*]} = -T$).

$A-normal$ if $A^{-1}T^*AT = TA^{-1}T^*A$ or equivalently, $T^{[*]}T = TT^{[*]}$.

$A-unitary$ if $T^*AT = A$ or equivalently, $T^{[*]} = T^{-1}$.

$A-hyponormal$ if $A(A^{-1}T^*AT - TA^{-1}T^*A) \geq 0$ or equivalently, if $A(T^{[*]}T - TT^{[*]}) \geq 0$.

II. BOUNDEDNESS AND ADJOINTS OF HILBERT SPACE OPERATORS

In what follows we briefly describe the concept of bounded linear operators using already known results and illustrate boundedness with a specific example of an integral operator. The following definitions, remarks and Theorems are required:

Definition 2.1: The graph of a linear operator $T: H \longrightarrow K$ is the set given by $\mathcal{G}(T) = \{(x, Tx): x \in D(T)\}$. This is a linear subspace of the Hilbert space $H \oplus K$ that has the full information about the operator T.

The graph norm of T is the scalar product defined on the domain $D(T)$ and is given by

$\|x\|_T = (\|x\| + \|Tx\|^2)^{1/2}, x \in D(T)$.

Definition 2.2: An operator T is called closed if its graph $\mathcal{G}(T)$ is a closed subset of $H \oplus K$ and T is closable if there exists a closed linear operator S from H to K such that $T \subseteq S$.

For a linear operator $T: H_1 \longrightarrow H_2$, its domain is denoted by $D(T)$ and is basically a subspace of H_1. To describe the adjoint of a linear operator T in a Hilbert space(s), consider a dense domain $D(T)$ in H_1 by setting

$$D(T^*) = \{y \in H_2: \exists\ x \in H_1, \langle Tx, y\rangle_2 = \langle x, u\rangle_1\}.$$

Using the $Riesz'\ Theorem$ [2], $y \in H_2$ is an element of $D(T^*)$ exactly when $x \longrightarrow \langle Tx, y\rangle_2$ is a continuous linear functional on $(D(T), \|.\|)$.

Because $D(T)$ is dense in H_1, the vector $u \in H_1$ satisfying $\langle Tx, y\rangle_2 = \langle x, u\rangle_1$ for all $x \in D(T)$ is uniquely determined by y.

Thus a linear transformation $T^*: H_2 \longrightarrow H_1$ which is well defined and linear can be obtained by settin $T^*y = u$.

T^* is called the adjoint of T which is simply defined by the equality $\langle Tx, y\rangle_2 = \langle x, T^*y\rangle_1$ $\forall\ x \in D(T), y \in D(T^*)$. In particular, if $T = T^*$ we say that T is $self-adjoint$ and $essentially\ self-adjoint$ if its closure $\bar{T}$ is self-adjoint, that is the operator $\bar{T}$ is called the closure of the closable operator T ([5] Definition 3.1.2). Some slight formulation of closed and closable operators are illustrated in Konrad ([5], Propositions 1.4 and 1.5). Other properties of the adjoint of an operator have been extensively studied by Huston et al [2].

Remark 2.3: The adjoint in a Hilbert space for an operator $T \in B(H)$ and a given scalar $\alpha \in \mathbb{C}$, will be defined as $(\alpha T)^* = \bar{\alpha}T^*$ where we use the fact that $\langle x, \alpha y\rangle = \bar{\alpha}\langle x, y\rangle\ \forall x, y \in H$.

We shall also require the following results to describe the adjoint of an operator:

Definition 2.4: If H and K are Hilbert spaces, a *sesquilinear form* $u: H \times K \longrightarrow \mathbb{C}$ is a transformation that satisfies the following properties:

1. $u(\alpha h + \beta g, k) = \alpha u(h,k) + \beta u(g,k)$ and
2. $u(h, \alpha k + \beta f) = \bar{\alpha} u(h,k) + \bar{\beta} u(h,f)\ \forall\, h, g \in H$ and all $k, f \in K$ and all scalars α and β.

Further, it has to be noted that a sesquilinear form u is bounded if there exists a finite constant M such that $|u(h,k)| \leq M \parallel h \parallel \parallel k \parallel\ \ \forall\ h, g \in H$ and $k \in K$. The following theorem describes all bounded sesquilinear forms:

Theorem 2.5: Let H and K be Hilbert Spaces and suppose that $u: H \times K \longrightarrow \mathbb{C}$ is a bounded sesquilinear form. Then there exists a unique $A \in B(H,K)$ such that $u(h,k) = \langle Ah, k \rangle_K\ \ \forall\, h \in H$ and $k \in K$.
As a consequence of this theorem, if $A \in B(H,K)$ and then define $u: H \times K \longrightarrow \mathbb{C}$ by $u(k,h) = \langle k, Ah \rangle_K$ (which is a bounded sesquilinear form) we obtain a unique operator
$A^* \in B(H,K)$ that satisfies the equation $u(k,h) = \langle A^* k, h \rangle_K\ \forall\ k \in K$ and $h \in H$. Taking conjugates leads us to the following result.

Theorem 2.6: Given Hilbert Spaces H and K and $A \in B(H,K)$ there exists a unique $A^* \in B(H,K)$ such that $\langle Ah, k \rangle_K = \langle h, A^* k \rangle_H\ \forall\, h \in H$ and $k \in K$. We call A^* the Hilbert space adjoint of A. In particular, if $H = K$ and $A^* = A$ the operator A is said to be *self − adjoint* or *Hermitian*.
Orthogonal projections onto closed subspaces are some of the examples of self-adjoint operators. A generalized result on self-adjointness of bounded linear operators is as follows:

Theorem 2.7: Let $T \in B(H)$. Then

a) If T is self-adjoint, $\langle Tx, x \rangle$ is real for all $x \in H$.

b) If H is complex and $\langle Tx, x \rangle$ is real for all $x \in H$, then the operator T is self-adjoint.

Remark 2.8: The products (composites) and sequences of self-adjoint operators in many instances appear in applications of analysis. Related generalizations for these are seen in ([6], Theorems 3.10.4 and 3.10.5) and they also outline basic ideas about the properties of self-adjoint operators. These results are also helpful in the analysis of their spectral pictures.

Example 2.9: The adjoint of an *integral operator* can be illustrated by considering a σ−finite measure space $(X, \mathfrak{M}, \mu)$ and a measurable function $k: X \times X \longrightarrow \mathbb{C}$ with $k \in L^2(X \times X, \mu \times \mu)$. Then the mapping $K: L^2(X,\mu) \longrightarrow L^2(X,\mu)$ is the integral operator defined by $Kf = g$ where $g(x) = \int_X k(x,y) f(y) d\,\mu(y)$ for $x \in X$. k is the kernel of the integral operator K. We determine the adjoint of K by finding an operator K^* such that $\langle Kf, g \rangle = \langle f, K^* g \rangle\ \forall\ f, g \in L^2(X,\mu)$. Writing the inner product as an integral and using *Fubini's Theorem* to interchange the order of integration, we have

$$\begin{aligned}\langle Kf, g \rangle &= \int_X Kf(x)\overline{g(x)}\, d\,\mu(x)\\ &= \int_X \left(\int_X k(x,y)\overline{g(x)}\, d\,\mu(x) \right) f(y)\, d\mu(y)\\ &= \int_X \left(\int_X \overline{\overline{k(x,y)}\, g(x)\, d\mu(x)} \right) f(y)\, d\mu(y)\\ &= \int_X f(y)\overline{K^* g(Y)}\, d\,\mu(y) = \langle f, K^* g \rangle,\end{aligned}$$

that is K^* is the adjoint of the integral operator K with kernel $k^*(x,y) \equiv \overline{k(y,x)}$. In other terms, K is self adjoint if and only if its kernel is Hermitian, that is $k(x,y) \equiv \overline{k(y,x)}$ or in the real case symmetric, that is $k(x,y) \equiv k(y,x)$.

III. UNITARY OPERATORS AND A- UNITARY EQUIVALENCE OF OPERATORS

In what follows, we describe the relationship between A- self-adjoint operators, unitary equivalence and A-unitarily equivalent of operators on a Hilbert space H. It is well known that unitary equivalence is an equivalence relation. It has also to be noted that unitary equivalence need not preserve A-self-adjointness of operators in general.

Definition 3.1: Recall also that two linear operators $T \in B(H)$ and $S \in B(K)$ are said to be A −unitary equivalent (denoted $T \overset{A}{\cong} S$), if there exists an *A − unitary* operator $U \in \mathcal{G}(H,K)$ such that

$TU = US$,that is, $TU = US$ and $U^*AU = A$, that is $U^{[*]} = U^{-1}$.

Remark 3.2: Every A-unitary operator T is invertible. We note that if T is A-unitary then T^* is also A-unitary. This follows from the fact that $(T^{[*]})^* = (T^{-1})^* = (T^*)^{[*]}$ $\Rightarrow (T^*)^{[*]} = (T^*)^{-1} \Rightarrow T^*$ is A-unitary.

The following result, however gives a condition which ensures that unitary equivalence preserves A-self adjointness of operators.

Theorem 3.4 ([12], Theorem 5.2): Let $T \in B(H)$ be an A-unitary operator .Then T is invertible and similar to $(T^{-1})^*$.

Proof: Since T is A-unitary , we have $T^*AT = A$. Since A is invertible, so is T^*AT and T^*. Invertibility of T^* implies invertibility of T and $(T^*)^{-1} = (T^{-1})^*$.To see this, note that for any $x, y \in H$, we have that

$\langle y, (A^{-1})^* A^* x \rangle = \langle A^{-1}y, A^*x \rangle = \langle AA^{-1}y, x \rangle = \langle y, x \rangle$ and

$\langle y, A^* (A^{-1})^* x \rangle = \langle Ay, (A^{-1})^* x \rangle = \langle A^{-1}Ay, x \rangle = \langle y, x \rangle$ and we conclude that

$(A^{-1})^* A^* x = A^* (A^{-1}) x = x$; and hence $(T^*)^{-1} .= (T^{-1})^*$.Computations reveal that

$T = A^{-1} (T^{-1})^* A$ as required.

Theorem 3.5 ([12], Theorem 5.3): Let $T \in B(H)$ be an A-unitary operator .Then the eigenvalues α of T are unimodular(i.e $|\alpha|^2 = 1$) or they come in inverse complex conjugate pairs $\left\{\alpha, \frac{1}{\alpha}\right\}$ or $\left\{\bar{\alpha}, \frac{\bar{1}}{\alpha}\right\}$. This means that the eigenvalues of T come in quartets $\lambda\ \frac{1}{\lambda}, \bar{\lambda}, \frac{\bar{1}}{\lambda}$.

Proof: Suppose $Tv = \alpha v$ where $\alpha \in \mathbb{C}$ and $v \in H$. Then

$$\langle Av, v \rangle = \langle T^*ATv, v \rangle \text{ ie}$$
$$\langle A T^{-1}v, v \rangle = \langle T^*ATv, v \rangle \text{ ie,}$$
$$\frac{1}{\alpha}\langle Av, v \rangle = \langle T^*Av, v \rangle.$$

Since A is invertible, $0 \notin \sigma(A)$ and $Av \neq 0$ for $v \neq 0$. Thus, $T^*Av = \frac{1}{\alpha}Av$ which means that $\frac{1}{\alpha}$ is an eigen value of T^* corresponding to the eigenvector Av.But eigenvalues of T^* are complex conjugates of those of T. Therefore $\frac{1}{(\bar{\alpha})}$ is an eigenvalue of T. We thus have $\alpha = \frac{1}{(\bar{\alpha})}$ which implies that $|\alpha|^2 = 1$ and hence $|\alpha| = \pm 1$ or $\left\{\alpha, \frac{1}{(\bar{\alpha})}\right\}$ is a pair of distinct inverse complex-conjugate eigenvalues of T.

The following results will enable us establish the relationship between A-unitary equivalent and A-normal operators.

Remark 3.6: The automorphism group of A-unitary operators is the set $\mathbb{G}_A = \{T \in B(H) : T^{[*]} = T^{-1}\}$.

Theorem 3.7 ([3], Theorem 4.3): Every unitary operator is A –unitary.

Theorem 3.8: For bounded linear operators on a Hilbert space H, i.e $S, T, R \in B(H)$, A –unitary equivalence is an equivalence relation .

Proof: First, recall that two linear operators $S, T \in B(H)$ A –unitarily equivalent (denoted $S \overset{A}{\cong} T$), if there exists an A –unitary operator U such that $TU = US$ .That is, $TU = US$ and $U^*AU = A$. This is equivalent to saying that $T = USU^{-1}$ and $U^*AU = A$. Clearly, $T \overset{A}{\cong} T$. To see this, simply let $U = I$. This proves that the relation $\overset{A}{\cong}$ is reflexive.

Suppose that $T \overset{A}{\cong} S$. Then by definition, there exists an A- unitary operator U such that $T = USU^{-1}$ and $U^*AU = A$ .Rewriting gives $S = U^{-1}TU$ and $U^*AU = A$. This proves that $\overset{A}{\cong} T$. Thus the relation $\overset{A}{\cong}$ is symmetric.

Now suppose that $R \in B(H)$ and that $T \overset{A}{\cong} S$ and $S \overset{A}{\cong} R$. Then by definition, there exist two A- unitary operators U and V such that $T = USU^{-1}$ and $U^*AU = A$ and $S = VRV^{-1}$ and $V^*AV = A$. A simple computation shows that

$$T = USU^{-1} \text{ and } U^*AU = A$$
$$= UVRV^{-1}U^{-1}, U^*AU = A \text{ and } V^*AV = A$$
$$= UVR(UV)^{-1}, U^*AU = A \text{ and } V^*AV = A \qquad (*).$$

From $(*)$, we have that $U^*AU = A$ .Premultiplication by V^* and post-multiplication by V both sides we get

$V^*U^*AUV = V^*AV = A$. That is $(UV)^*A(UV) = A$.

Now let $W = UV$. By ([12], Theorem 5.2), every A-unitary operator is invertible and similar to the adjoint of its inverse. From ([12], Theorem 5.8), the product of A-unitary operators is A-unitary. From the remark immediately after ([12], Theorem 5.1), if an operator is A-unitary, then its adjoint is also A-unitary. Combining these results we conclude that W is also A-unitary. Thus, $(*)$ becomes $T = WRW^{-1}$ and $W^*AW = A$. This proves that $T \overset{A}{\cong} R$. Therefore $\overset{A}{\cong}$ is transitive. This proves the claim.

Remark 3.9: In view of Theorem 3.8, it is evident that A-unitary equivalence is stronger than similarity, but weaker than unitary equivalence for operators in a Hilbert space. That is

Unitary equivalence $\Longrightarrow$ A-unitary equivalence$\Longrightarrow$Similarity.

The following results are well known and are needed in the sequel.

Theorem 3.10 (Nzimbi, et al [12]): If T is a normal operator and $S \in B(H)$ is unitarily equivalent to T, then S is normal.

Theorem 3.11 (Nzimbi, et al [12]): Every normal operator T is A-normal.

The following result shows that A-unitary equivalence preserves A-normality of operators.

Corollary 3.12: Let $A = A^* = A^{-1}$ **.** If T is an A- normal operator and $S \in B(H)$ is A- unitary equivalent to T then S is A-normal.

Proof: By definition of A-unitary equivalence of operators $T \in B(H)$ and $S \in B(H)$, $S = XTX^{-1}$ and $X^*AX = A$. Since T is A -normal , we have that $A^{-1}(T^*A)T = TA^{-1}T^*A$ or equivalently, $T^*AT = A(TA^{-1}T^*)A$. To prove that S is A-normal we need to prove that $S^*A\ S = A(SA^{-1}S^*)A$. This follows from the fact that

$$S^*AS = [(X^{-1})^*T^*X^*]A[XTX^{-1}]$$

$= (X^{-1})^*(T^*AT)X^{-1}$ (since $X^*AX = A$)

$= (X^{-1})^*[A(TA^{-1}T^*)A\ ]X^{-1}$ (since $T^*AT = A(TA^{-1}T^*)A$)

$= AX(TA^{-1}T^*)AX^{-1}$ (since $(X^{-1})^*A = AX$)

$= A(XT)A^{-1}T^*AX^{-1}$

$= A(XT)A^{-1}T^*X^*A$ (since $X^*A = AX^{-1}$)

$= A(SX)A^{-1}(T^*X^*)A$ (since $XT = SX$)

$= A(SX)A^{-1}(X^*S^*)A$ (since $T^*X^* = X^*S^*$)

$= A(SA^{-1}S^*)A$ (since $X^*A = AX^{-1}$). This proves that S is A-normal as required.

IV. DESCRIBING A-UNITARY OPERATORS USING POLAR DECOMPOSITIONS.

An operator T is said to be a quasi-isometry if $T^{*2}T^2 = T^*T$, that is, if it is a T^*T $-$isometry. (See more results in [12], [13] and [18]). In view of this it clear that every isometry is a quasi-isometry, however the converse is not true. It has also been shown in [12] that a quasi-isometry which is an m-isometry is an isometry. Further, it has been proved that any quasi-isometry T its norm is unity is hyponormal.

Proposition 4.1([12], Proposition 5.11): Let T be an invertible quasinormal operator on a Hilbert Space H and let $T = UT$ be the polar decomposition of T. Then U is a T^*T $-$unitary.

Corollary 4.2 ([18], Corollary 4.4.3). Let T be a hyponormal contraction on H with $\| T \| = 1$. Then T is a normal isometry exactly when T^* is a T^*T $-$unitary.

Using these two results together with other results proved in [11] and [18], the following results can also be proved:

Theorem 4.3: Let T be an invertible self-adjoint quasi-isometry and suppose $\| T \| = 1$. Then T is a normal if and only if T^* is a T^*T $-$unitary.

Proof: Suppose $T = UT$ is the polar decomposition of T. Since T is a quasi-isometry of norm one, by ([11], Theorem 2.2), T is hyponormal. If T is normal, then by ([18], Corollary 4.3.8) it is a partial isometry. Again by ([11], Theorem 2.3) T is quasi normal. Hence the result follows from Corollary 4.2 above.

Proposition 4.4: Let $T \in B(H)$ be A-unitary.Then $\overline{Ran\ (A)}$ reduces T and $T|_{\overline{Ran\ (A)}}$ is A-unitary.

Proof: First, note that $Ran(T^*AT) = Ran(A)$. Since A is self-adjoint and invertible, we have $Ker\ (A) = Ker(A^*) = \{0\}$ and hence $Ran\ (T^*) = Ran\ (A) = H$.This means that $Ker\ (H)^{\perp} = H$ and hence $Ker(T) = \{0\}$. Note that $Ran\ (A) = T^*Ran(AT) = T^*(Ran(A))$, which shows that $Ran\ (A)$ is T^* $-$invariant.Note also that $Ker\ (A^*) = H \bigoplus \overline{Ran\ (A)}$.

Remark 4.5: It has been shown in ([12], Proposition 5.15) that if T is skew-adjoint, then e^{Tt} is unitary for all $t \in \mathbb{R}$.This claim is also true if T is self-adjoint. If T is self-adjoint, then the operator e^{-iTt} is unitary for all $t \in \mathbb{R}$. However, this is not the case if T is skew-adjoint .We now illustrate that an interesting relationship between A-skew adjoint operators and A-unitary operators in the following Theorem.

Theorem 4.6: Let T A-skew-adjoint.Then the time evolution operator $E(t) = e^{-iTt}$ is A-unitary but not unitary.
Proof: By computation, it is established that $E^*AE = A$ meaning that the time evolution operator is A-unitary. However, $E^*E = EE^* \neq I$.This completes the proof.

Remark 4.7: Results have been shown that the principal square root $T^{1/2}$ of T exists exactly when the operator T has no eigenvalues on $\mathbb{R}^- = \{ x \in \mathbb{R}: x \leq 0\}$ and that $T^{1/2}$ is unique whenever it exists. If this is the case, we can write $T = T^{1/2}T^{1/2}$ (See related results in [12], Lemma 5.20 and Theorem 5.21).

Theorem 4.8: Suppose T is A-unitary.In the polar decomposition $T = U|T|$ the factors U and $|T|$ are A-unitary .
Proof: Since T is A-unitary, T is invertible and so U and $|T|$ exist and are unique. We have T A-unitary if and only if $T = T^{-1[*]} = (U|T|)^{-1[*]} = U^{-1[*]}|T|^{-1[*]}$. Clearly, $U^{-1[*]}$ is unitary and $|T|^{-1[*]}$ is self-adjoint and positive.Thus $T = U^{-1[*]}|T|^{-1[*]}$ is another polar decomposition of . Uniqueness of polar factors implies that $U = U^{-1[*]}$ and $|T| = |T|^{-1[*]}$. Thus U and $|T|$ are A-unitary as required.

Example 4.9: Consider the skew -adjoint matrix $T = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. This matrix has polar decompositions

$T = U_1|T| = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, where U_1 is not A-skew- adjoint ,

$T = U_2|T| = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, where U_2 is A-skew- adjoint and

$T = U_3|T| = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, where U_3 is A-skew- adjoint.

By these decompositions, it is evident that the set of invertible matrices is dense in the class of A-skew- adjoint operators. Thus a singular A-skew- adjoint operator may have polar decomposition with an A-skew- adjoint or non- A-skew- adjoint unitary factor U as well as having more than one polar decomposition, with the unitary factor being A-skew- adjoint.

Theorem 4.10([12], Theorem 5.23): Suppose T is either A-skew- adjoint or A-self- adjoint.Then $|T|$ is self-adjoint if and only if T is normal.
Proof: $|T| = \sqrt{T^*T}$ A-self- adjoint implies that T^*T is A-self- adjoint. That is $(T^*T)^{[*]} = T^*T$. Thus, $T^*T = (T^*T)^{[*]} = T^{[*]}(T^*)^{[*]} = T^{[*]}(T^{[*]})^* = TT^*$.
But T A-skew- adjoint means that $T^{[*]} = -T$, so
$T^*T = T^{[*]}(T^*)^{[*]} = (-T)(-T^*) = TT^*$, which proves that T is normal.
Conversely, we note that if T is A-skew- adjoint , then so is T^*. Since T is normal, we have T^*T is A-self-adjoint. Thus, $|T| = \sqrt{T^*T}$ is A-self –adjoint by ([12], Theorem 5.21).
Corollary 4.11: Suppose T is A-unitary, with A an invertible, self-adjoint and involutory(i.e. A is a symmetry, $A^* = A = A^{-1}$). Then $\pm 1 \notin \sigma_r(T)$.
Proof: Clearly, from the claim that $\pm 1 \in \sigma_p(T)$, the result follows immediately, since $\sigma_p(T) \cap \sigma_r(T) = \emptyset$.

V. CONCLUSION

a) A-unitary equivalence is stronger than similarity, but weaker than unitary equivalence for operators in a Hilbert space. That is

Unitary equivalence $\Rightarrow$ A-unitary equivalence$\Rightarrow$Similarity.

b) An A-skew-adjoint operator is A-unitary but not unitary.

c) Every normal operator T is A-normal. However, it has to be noted that there exist A-normal operators which are not normal. For example, if $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} i & i \\ i & 0 \end{pmatrix}$, computations show that $T^{[*]}T = TT^{[*]}$ and $T^*T \neq TT^*$. Therefore, T is A-normal but not normal.
In fact, A-self-adjoint, A-skew-adjoint and A-unitary operators are special cases of A-normal operators.

d) It has been established that every hyponormal operator is A-hyponormal. Moreover, every A-skew-adjoint , A-unitary operators and A-normal operator T is A-hyponormal.(See [12], Theorems 6.1,6.2 and 6.3).We thus have the following class inclusions:

$$Symmetry \subsetneq Unitary \subsetneq Normal \subsetneq A-Normal \subsetneq A-Hyponormal$$ and

$$Symmetry \subsetneq Self-adjoint \subsetneq Normal \subsetneq A-Normal \subsetneq A-Hyponormal.$$

$$Skew-adjoint \subsetneq Normal \subsetneq A-Normal \subsetneq A-Hyponormal$$

In addition the intersection of the class of self-adjoint and unitary operators yields a symmetry, i.e.

$$\{Self-adjoint\} \cap \{Unitary\} = \{Symmetry\}.$$

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