On Relative Order of Composite Function with respect to a Composite Function

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ABSTRACT

In this paper we prove some results on relative order of composite function with respect to a composite function formed with entire and meromorphic functions.

Keywords: *Entire function, Meromorphic function, Relative order, Composite function.*

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I. INTRODUCTION AND DEFINITIONS

The Maximum modulus of an entire function $f(z)$ is defined by

$$M_f(r) = max\{|f(z)| : |z| = r\}.$$

If f is non-constant then $M_f(r)$ is strictly increasing and continuous function of r and its inverse

$M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$

exists and is such that

$\lim_{r\to\infty} M_f(r) = \infty.$

Definition 1.1 The order of an entire function $f(z)$ is defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}.$$

Definition 1.2([4])If $f(z)$ and $g(z)$ are two entire functions then the relative order of $f(z)$ with respect to $g(z)$ is defined as

$$\rho_g(f) = \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \quad for\ all\ r > r_0(\mu) > 0\}$$
$$= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

Definition 1.3 ([7]) The relative order of a meromorphic function $f(z)$ with respect to an entire function $g(z)$ is defined as

$$\rho_g(f) = \inf\{\lambda > 0 : T_f(r) < T_g(r^\lambda) \quad for\ all\ r > r_0(\lambda) > 0\}$$
$$= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

Definition 1.4([1]) The relative order of a meromorphic function $f(z)$ with respect to another meromorphic function $g(z)$ is defined as

$$\rho_g(f) = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log T_h(r)}.$$

II. LEMMAS

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1([6]) Let g be an entire function. Then for all large values of r

$$T_g(r) \le \log M_g(r) \le 3T_g(2r).$$

Lemma 2.2([12]) Let f and g be two entire functions. Then for a sequence of values of r tending to infinity

$$T_{f\circ g}(r) \ge \frac{1}{3}\log M_f(\frac{1}{9}M_g(\frac{r}{4})).$$

Lemma 2.3([5]) Let f and g be two entire functions. Then for all sufficiently large values of r

$$M_f(\frac{1}{8}M_g(\frac{r}{2}) - |g(0)|) \le M_{f\circ g}(r) \le M_f(M_g(r)).$$

Lemma 2.4([8]) Let f be a meromorphic function and g be an entire function with $0<\mu<\rho_g<\infty$ and $\lambda_f > 0.$ Then for a sequence of values of r tending to infinity

$$T_{f\circ g}(r) \ge T_g(\exp(r)^{\mu}).$$

Lemma 2.5 ([3]) Let f be a meromorphic function and g be an entire function with $0<\mu<\rho_g \le \infty.$ Then for a sequence of values of r tending to infinity

$$T_{f\circ g}(r) \ge T_f(\exp(r)^{\mu}).$$

Lemma 2.6 ([4]) Let f be a meromorphic function and g be an entire function then for all large values of r

$$T_{f\circ g}(r) \le \{1+0(1)\}\frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)).$$

Lemma 2.7([9]) Let f and g be two entire functions. If $M_g(r) > \frac{2+\epsilon}{\epsilon}|g(0)|$ for any $\epsilon > 0,$ then

$$T_{f\circ g}(r) < (1+\epsilon)T_f(M_g(r)).$$

In particular if $g(0)=0$ then

$$T_{f\circ g}(r) < T_f(M_g(r))$$

for all $r>0.$

Lemma 2.8 Let f be a meromorphic function and g be an entire function with $0 \le \rho_g < \mu < \infty.$ Then for all large values of r

$$T_{f\circ g}(r) \le \{1+0(1)\}T_f(\exp(r)^{\mu}).$$

Proof: From Lemma 2.6

$$T_{f\circ g}(r) \le \{1+0(1)\}\frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)). \quad (2.1)$$

Now from the definition of order of g we have for any $\epsilon > 0$ and large r

$$\log M_g(r) < r^{\rho_g + \epsilon}$$

$$i.e,\ M_g(r) < \exp r^{\rho_g+\epsilon} < \exp r^{\mu} \quad when \quad \mu > \rho_g. \text{(2.2)}$$

So using Lemma 2.1we have from (2.1) and (2.2)

$$T_{f\circ g}(r) \le \{1+0(1)\}T_f(\exp(r)^{\mu}).$$

III. MAIN THEOREMS

In this section we present the main results of the paper.

Theorem 3.1*Let* f_1, f_2, h_1 *and* h_2 *be four entire functions of respective finite orders and* g *be a polynomial of degree* m. *Then the relative order of* $h_1 \circ h_2$ *with respect to* $f_1 \circ f_2 \circ g$ *satisfies the inequality*

$$\rho_{f_1\circ f_2\circ g}(h_1 \circ h_2) \ge \frac{\lambda_{h_2}}{m\rho_{f_2}}$$

and further when $|h_2(0)| = 0,$

$$\rho_{f_1\circ f_2\circ g}(h_1 \circ h_2) \le \frac{\rho_{h_2}}{m\lambda_{f_2}}.$$

Proof: We have by the definition of order for any $\epsilon > 0$ there exists $r_0(\epsilon) > 0$ such that

$$M_{f_1}(r) \le \exp\{r^{\rho_{f_1}+\epsilon}\} \quad for\ \ all\ \ r > r_0(\epsilon). \tag{3.1}$$

$$i.e,\ M_{f_1}^{-1}(r) > \exp\{\frac{1}{\rho_{f_1}+\epsilon}\log^{[2]} r\}. \tag{3.2}$$

Similarly

$$M_{f_2}^{-1}(r) > \exp\{\frac{1}{\rho_{f_2}+\epsilon}\log^{[2]} r\}. \tag{3.3}$$

Again for arbitrary $\epsilon > 0$ and for all large values of r

$$M_{h_1}(r) > \exp\{r^{\lambda_{h_1}-\epsilon}\} \tag{3.4}$$

and

$$M_{h_2}(r) > \exp\{r^{\lambda_{h_2}-\epsilon}\}. \tag{3.5}$$

Let $g(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m.$

Then for any $\epsilon > 0$ there exists $r_1(\epsilon) > 0$ such that

$$|a_m| r^m(1-\epsilon) < M_g(r) < |a_m| r^m(1+\epsilon) \quad for\ \ all\ \ r > r_1. \tag{3.6}$$

So

$$M_g^{-1}(r) > \{\frac{r}{|a_m|(1+\epsilon)}\}^{\frac{1}{m}} \ \ and \ \ M_g^{-1}(r) < \{\frac{r}{|a_m|(1-\epsilon)}\}^{\frac{1}{m}}. \qquad (3.7)$$

Again from the Lemma 2.3we get

$$M_{f_1\circ f_2\circ g}(r) < M_{f_1}(M_{f_2}(M_g(r)))$$

$$i.e, \ M^{-1}_{f_1\circ f_2\circ g}(r) > M_g^{-1}(M_{f_2}^{-1}(M_{f_1}^{-1}(r))). \ (3.8)$$

Also by Lemma 2.3for two entire functions h_1 and h_2 with $|h_2(0)|=0$ we have

$$M_{h_1\circ h_2}(r) \geq M_{h_1}(\frac{1}{8}M_{h_2}(\frac{r}{2})). \ (3.9)$$

So

$$\rho_{f_1\circ f_2\circ g}(h_1\circ h_2) = \limsup_{r\to\infty}\frac{\log M^{-1}_{f_1\circ f_2\circ g}(M_{h_1\circ h_2}(r))}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}(M_{f_2}^{-1}(M_{f_1}^{-1}(M_{h_1\circ h_2}(r))))}{\log r} \ \ from \ (3.8)$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}(M_{f_2}^{-1}(M_{f_1}^{-1}(M_{h_1}(\frac{1}{8}M_{h_2}(\frac{r}{2})))))}{\log r} \ \ from \ (3.9)$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}(M_{f_2}^{-1}(M_{f_1}^{-1}(M_{h_1}(\frac{1}{8}\exp(\frac{r}{2})^{\lambda_{h_2}-\epsilon}))))}{\log r} \ \ from \ (3.5)$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}(M_{f_2}^{-1}(M_{f_1}^{-1}(\exp[\frac{1}{8}\exp(\frac{r}{2})^{\lambda_{h_2}-\epsilon}]^{\lambda_{h_1}-\epsilon})))}{\log r} \ \ from \ (3.4)$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}(M_{f_2}^{-1}(\exp\{\frac{1}{\rho_{f_1}+\epsilon}\log^{[2]}(\exp[\frac{1}{8}\exp(\frac{r}{2})^{\lambda_{h_2}-\epsilon}]^{\lambda_{h_1}-\epsilon})\}))}{\log r} \ \ from \ (3.2)$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}(M_{f_2}^{-1}(\exp\{\frac{\lambda_{h_1}-\epsilon}{\rho_{f_1}+\epsilon}(\frac{r}{2})^{\lambda_{h_2}-\epsilon}\}))+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}(\exp\{\frac{1}{\rho_{f_2}+\epsilon}\log^{[2]}(\exp\{\frac{\lambda_{h_1}-\epsilon}{\rho_{f_1}+\epsilon}(\frac{r}{2})^{\lambda_{h_2}-\epsilon}\}))+O(1)}{\log r} \ \ from \ (3.3)$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}(\exp\{\frac{\lambda_{h_2}-\epsilon}{\rho_{f_2}+\epsilon}\log(\frac{r}{2})\})+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty}\frac{\log M_g^{-1}((\frac{r}{2})^{\frac{\lambda_{h_2}-\epsilon}{\rho_{f_2}+\epsilon}})+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty} \frac{\log\{\frac{(\frac{r}{2})^{\frac{\lambda_{h_2}-\epsilon}{\rho_{f_2}+\epsilon}}}{|a_m|(1+\epsilon)}\}^{\frac{1}{m}}+O(1)}{\log r} \quad from \ (3.7)$$

$$\geq limsup_{r\to\infty} \frac{1}{m}\frac{\log(\frac{r}{2})^{\frac{\lambda_{h_2}-\epsilon}{\rho_{f_2}+\epsilon}}-\log|a_m|(1+\epsilon)+O(1)}{\log r}$$

$$\geq limsup_{r\to\infty} \frac{1}{m}\frac{\frac{\lambda_{h_2}-\epsilon}{\rho_{f_2}+\epsilon}\log(\frac{r}{2})-\log|a_m|(1+\epsilon)+O(1)}{\log r}$$

$$\geq \frac{1}{m}\frac{\lambda_{h_2}-\epsilon}{\rho_{f_2}+\epsilon}$$

$$\geq \frac{1}{m}\frac{\lambda_{h_2}}{\rho_{f_2}}, \ since \ \epsilon>0 \ is \ arbitrary.$$

From Lemma 2.3, for all sufficiently large values of r

$$M_{f_1\circ f_2\circ g}(r)\geq M_{f_1\circ f_2}(\frac{1}{9}M_g(\frac{r}{2}))\geq M_{f_1}(\frac{1}{9}M_{f_2}(\frac{1}{18}(M_g(\frac{r}{2}))))\tag{3.10}$$

$$i.e, \ M^{-1}_{f_1\circ f_2\circ g}(r)\leq 2M_g^{-1}(18M_{f_2}^{-1}(9M_{f_1}^{-1}(r))).\tag{3.11}$$

Also for all sufficiently large values of r we get from Lemma 2.3

$$M_{h_1\circ h_2}(r)\leq M_{h_1}(M_{h_2}(r)).\tag{3.12}$$

Againfor sufficiently large values of r

$$M_{h_1}(r)\leq \exp\{r^{\rho_{h_1}+\epsilon}\} \quad and \quad M_{h_2}(r)\leq \exp\{r^{\rho_{h_2}+\epsilon}\}\tag{3.13}$$

and

$$M_{f_1}^{-1}(r)<\exp\{\frac{1}{\lambda_{f_1}-\epsilon}\log^{[2]}r\} \quad and \ M_{f_2}^{-1}(r)<\exp\{\frac{1}{\lambda_{f_2}-\epsilon}\log^{[2]}r\}.\tag{3.14}$$

Now

$$\rho_{f_1\circ f_2\circ g}(h_1\circ h_2)=\limsup_{r\to\infty}\frac{\log M^{-1}_{f_1\circ f_2\circ g}(M_{h_1\circ h_2}(r))}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\log(2M_g^{-1}(18M_{f_2}^{-1}(9M_{f_1}^{-1}(M_{h_1\circ h_2}(r)))))}{\log r} \quad from \ (3.11)$$

$$\leq limsup_{r\to\infty}\frac{\log(2M_g^{-1}(18M_{f_2}^{-1}(9M_{f_1}^{-1}(M_{h_1}(M_{h_2}(r))))))}{\log r} \quad from \ (3.12)$$

$$\leq limsup_{r\to\infty} \frac{\log(2M_g^{-1}(18M_{f_2}^{-1}(9M_{f_1}^{-1}(M_{h_1}(\exp\{r^{\rho_{h_2}+\varepsilon}\})))))}{\log r} \; from \;\; (3.13)$$

$$\leq limsup_{r\to\infty} \frac{\log(2M_g^{-1}(18M_{f_2}^{-1}(9M_{f_1}^{-1}(\exp[\exp\{r^{\rho_{h_2}+\varepsilon}\}]^{\rho_{h_1}+\varepsilon}))))}{\log r} \; from \;\; (3.13)$$

$$\leq limsup_{r\to\infty} \frac{\log(2M_g^{-1}(18M_{f_2}^{-1}(9\exp\{\frac{1}{\lambda_{f_1}-\varepsilon}\log^{[2]}(\exp[\exp\{r^{\rho_{h_2}+\varepsilon}\}]^{\rho_{h_1}+\varepsilon})\})))}{\log r} \; from \;\; (3.14)$$

$$\leq limsup_{r\to\infty} \frac{\log(2M_g^{-1}(18M_{f_2}^{-1}(9\exp\{\frac{\rho_{h_1}+\varepsilon}{\lambda_{f_1}-\varepsilon}r^{\rho_{h_2}+\varepsilon}\})))}{\log r}$$

$$\leq limsup_{r\to\infty} \frac{\log(2M_g^{-1}(18\exp\{\frac{1}{\lambda_{f_2}-\varepsilon}\log^{[2]}\{9\exp\{\frac{\rho_{h_1}+\varepsilon}{\lambda_{f_1}-\varepsilon}r^{\rho_{h_2}+\varepsilon}\}\}\}))}{\log r} \; from \;\; (3.14)$$

$$\leq limsup_{r\to\infty} \frac{\log(2M_g^{-1}(18\exp\{\frac{1}{\lambda_{f_2}-\varepsilon}\log[\frac{\rho_{h_1}+\varepsilon}{\lambda_{f_1}-\varepsilon}r^{\rho_{h_2}+\varepsilon}]\}))+O(1)}{\log r}$$

$$\leq limsup_{r\to\infty} \frac{\log(2M_g^{-1}(18r^{\frac{\rho_{h_2}+\varepsilon}{\lambda_{f_2}-\varepsilon}}))+O(1)}{\log r}$$

$$\leq limsup_{r\to\infty} \frac{\log\{\frac{18r^{\frac{\rho_{h_2}+\varepsilon}{\lambda_{f_2}-\varepsilon}}}{|a_m|(1-\varepsilon)}\}^{\frac{1}{m}}+O(1)}{\log r} \; from \;\; (3.7)$$

$$\leq limsup_{r\to\infty} \frac{\frac{1}{m}\log r^{\frac{\rho_{h_2}+\varepsilon}{\lambda_{f_2}-\varepsilon}}+O(1)}{\log r}$$

$$\leq \frac{1}{m}\frac{\rho_{h_2}}{\lambda_{f_2}}, \; since \; \varepsilon > 0 \;\; is \;\; arbitrary.$$

Theorem 3.2*Let* f_1, f_2, h_1 *and* h_2 *be four entire functions of respective finite orders such that* $\rho_{f_2} \neq 0$ *and* g_1, g_2 *be two polynomials of degree* m_1, m_2 *respectively such that* $|g_2(0)| = 0$. *Then the relative order of* $h_1 \circ h_2 \circ g_2$ *with respect to* $f_1 \circ f_2 \circ g_1$ *satisfies the inequality*

$$\rho_{f_1\circ f_2\circ g_1}(h_1 \circ h_2 \circ g_2) \geq \frac{m_2}{m_1}\frac{\lambda_{h_2}}{\rho_{f_2}}$$

and

$$\rho_{f_1\circ f_2\circ g_1}(h_1 \circ h_2 \circ g_2) \leq \frac{m_2}{m_1}\frac{\rho_{h_2}}{\lambda_{f_2}}.$$

Proof: Let $g_1(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{m_1} z^{m_1}$

and

$$g_2(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_{m_2} z^{m_2}$$

be two polynomials of degree m_1, m_2 respectively. By the definition of relative order of an entire function with respect to another entire function we have

$$\rho_{f_1\circ f_2\circ g_1}(h_1\circ h_2\circ g_2) = \limsup_{r\to\infty} \frac{\log M^{-1}_{f_1\circ f_2\circ g_1}(M_{h_1\circ h_2\circ g_2}(r))}{\log r}$$

$$\geq limsup_{r\to\infty} \frac{\log M_{g_1}^{-1}(M_{f_2}^{-1}(M_{f_1}^{-1}(M_{h_1\circ h_2\circ g_2}(r))))}{\log r} \quad from \ (3.8)$$

$$\geq limsup_{r\to\infty} \frac{\log M_{g_1}^{-1}(M_{f_2}^{-1}(M_{f_1}^{-1}(M_{h_1}(\frac{1}{9}M_{h_2}(\frac{1}{18}M_{g_2}(\frac{r}{2}))))))}{\log r} \quad from \ (3.10)$$

$$\geq limsup_{r\to\infty} \frac{\log M_{g_1}^{-1}(M_{f_2}^{-1}(M_{f_1}^{-1}(M_{h_1}(\frac{1}{9}M_{h_2}(\frac{1}{18}|b_m|(1-\epsilon)(\frac{r}{2})^{m_2})))))}{\log r} \quad from \ (3.6)$$

$$\geq limsup_{r\to\infty} \frac{\log M_{g_1}^{-1}(M_{f_2}^{-1}(M_{f_1}^{-1}(M_{h_1}(\frac{1}{9}\exp(\frac{1}{18}|b_m|(1-\epsilon)(\frac{r}{2})^{m_2}))^{\lambda_{h_2}-\epsilon})))}{\log r} \quad from \ (3.5)$$

$$\geq limsup_{r\to\infty} \frac{\log M_{g_1}^{-1}(M_{f_2}^{-1}(M_{f_1}^{-1}(\exp[\frac{1}{9}\exp(\frac{1}{18}|b_m|(1-\epsilon)(\frac{r}{2})^{m_2})^{\lambda_{h_2}-\epsilon}]^{\lambda_{h_1}-\epsilon})))}{\log r} \quad from \ (3.4)$$

$$\geq limsup_{r\to\infty} \frac{\log M_{g_1}^{-1}(M_{f_2}^{-1}(\exp\{\frac{\lambda_{h_1}-\epsilon}{\rho_{f_1}+\epsilon}\log(\frac{1}{9}\exp(\frac{1}{18}|b_m|(1-\epsilon)(\frac{r}{2})^{m_2})^{\lambda_{h_2}-\epsilon})\}))}{\log r} \quad from \ (3.2)$$

$$\geq limsup_{r\to\infty} \frac{\log M_{g_1}^{-1}(M_{f_2}^{-1}(\exp\{\frac{\lambda_{h_1}-\epsilon}{\rho_{f_1}+\epsilon}(\frac{|b_m|}{18})(1-\epsilon)(\frac{r}{2})^{m_2}\}^{\lambda_{h_2}-\epsilon})) + O(1)}{\log r}$$

$$\geq limsup_{r\to\infty} \frac{\log M_{g_1}^{-1}(\exp(\frac{1}{\rho_{f_2}+\epsilon}\log^{[2]}(\exp\{\frac{\lambda_{h_1}-\epsilon}{\rho_{f_1}+\epsilon}(\frac{|b_m|}{18})(1-\epsilon)(\frac{r}{2})^{m_2}\}^{\lambda_{h_2}-\epsilon}))) + O(1)}{\log r} \quad from \ (3.3)$$

$$\geq \limsup_{r\to\infty} \frac{\log M_{g_1}^{-1}(\exp\{\frac{\lambda_{h_2}-\epsilon}{\rho_{f_2}+\epsilon}\log(\{\frac{\lambda_{h_1}-\epsilon}{\rho_{f_1}+\epsilon}(\frac{|b_m|}{18})(1-\epsilon)(\frac{r}{2})^{m_2}\})\})+O(1)}{\log r}$$

$$\geq \limsup_{r\to\infty} \frac{\log M_{g_1}^{-1}(\exp\{\frac{\lambda_{h_2}-\epsilon}{\rho_{f_2}+\epsilon}\log(\frac{r}{2})^{m_2}\})+O(1)}{\log r}$$

$$\geq \limsup_{r\to\infty} \frac{\log M_{g_1}^{-1}[(\frac{r}{2})^{m_2}]^{\frac{\lambda_{h_2}-\epsilon}{\rho_{f_2}+\epsilon}}+O(1)}{\log r}$$

$$\geq \limsup_{r\to\infty} \frac{\frac{1}{m_1}\log\frac{[(\frac{r}{2})^{m_2}]^{\frac{\lambda_{h_2}-\epsilon}{\rho_{f_2}+\epsilon}}}{|a_m|(1+\epsilon)}+O(1)}{\log r} \quad from\ (3.7)$$

$$\geq \limsup_{r\to\infty} \frac{\frac{1}{m_1}\log(\frac{r}{2})^{m_2\frac{\lambda_{h_2}-\epsilon}{\rho_{f_2}+\epsilon}}+O(1)}{\log r} = \frac{m_2}{m_1}\frac{\lambda_{h_2}-\epsilon}{\rho_{f_2}+\epsilon}$$

$$\geq \frac{m_2}{m_1}\frac{\lambda_{h_2}}{\rho_{f_2}},\ since\ \epsilon>0\ is\ \ arbitrary.$$

Using the same arguments as in Theorem 3.1we can show that

$$\rho_{f_1\circ f_2\circ g_1}(h_1\circ h_2\circ g_2) \leq \frac{m_2}{m_1}\frac{\rho_{h_2}}{\lambda_{f_2}}.$$

Theorem 3.3*Let* f_1, f_2 *and* h_2 *be three entire functions of respective positive orders and* h_1 *be meromorphic function of finite order* ρ_{h_1} *and* g *be a polynomial of degree* m*. Then the relative order of* $h_1\circ h_2$ *with respect to* $f_1\circ f_2\circ g$ *satisfies the inequality*

$$\rho_{f_1\circ f_2\circ g}(h_1\circ h_2) \geq \frac{1}{m}\frac{\rho_{h_2}}{\rho_{f_2}}\ \ and\ \ \rho_{f_1\circ f_2\circ g}(h_1\circ h_2) \leq \frac{1}{m}\frac{\rho_{h_2}}{\lambda_{f_2}}.$$

Proof: For any $\epsilon>0$ and for all large values of r we get

$$T_{f_1}(r) < r^{\rho_{f_1}+\epsilon}\ \ and\ \ T_{h_1}(r) < r^{\rho_{h_1}+\epsilon}. \tag{3.15}$$

$$T_{f_1}^{-1}(r) > r^{\frac{1}{\rho_{f_1}+\epsilon}} \tag{3.16}$$

for all large values of r.

Also for all large values of r we have

$$T_{h_1}(r) > r^{\lambda_{h_1}-\epsilon}. \tag{3.17}$$

For three entire functions f_1, f_2 and h_2 we use the Lemma 2.7we get

$$T_{f_1\circ f_2\circ g}(r) \le 3T_{f_1}(M_{f_2}(M_g(r))).$$

So for all large values of r we get

$$T^{-1}_{f_1\circ f_2\circ g}(r) \ge M_g^{-1}(M_{f_2}^{-1}(T_{f_1}^{-1}(\frac{r}{3}))). \tag{3.18}$$

Again from Lemma 2.5we get for a sequence of values of r tending to infinity and $0<\mu<\rho_{h_2}$

$$T_{h_1\circ h_2}(r) \ge T_{h_1}(\exp(r)^{\mu}). \tag{3.19}$$

Let $g(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$.

Now

$$\rho_{f_1\circ f_2\circ g}(h_1\circ h_2) = \limsup_{r\to\infty} \frac{\log T^{-1}_{f_1\circ f_2\circ g}(T_{h_1\circ h_2}(r))}{\log r}$$

$$\ge limsup_{r\to\infty} \frac{\log M_g^{-1}(M_{f_2}^{-1}(T_{f_1}^{-1}(\frac{T_{h_1\circ h_2}(r)}{3})))}{\log r} \quad from \ (3.18)$$

$$\ge limsup_{r\to\infty} \frac{\log M_g^{-1}(M_{f_2}^{-1}(T_{f_1}^{-1}(\frac{T_{h_1}(\exp r^{\mu})}{3})))}{\log r} \quad from \ (3.19)$$

$$\ge limsup_{r\to\infty} \frac{\log M_g^{-1}(M_{f_2}^{-1}(T_{f_1}^{-1}(\{\exp r^{\mu}\}^{\lambda_{h_1}-2\epsilon})))}{\log r} \quad from \ (3.17)$$

$$\ge limsup_{r\to\infty} \frac{\log M_g^{-1}(M_{f_2}^{-1}([\{\exp r^{\mu}\}^{\lambda_{h_1}-2\epsilon}]^{\frac{1}{\rho_{f_1}+\epsilon}}))}{\log r} \quad from \ (3.16)$$

$$\ge limsup_{r\to\infty} \frac{\log M_g^{-1}(M_{f_2}^{-1}([\{\exp r^{\mu}\}]^{\frac{\lambda_{h_1}-2\epsilon}{\rho_{f_1}+\epsilon}}))}{\log r}$$

$$\ge limsup_{r\to\infty} \frac{\log M_g^{-1}(\exp\{\frac{1}{\rho_{f_2}+\epsilon}\log^{[2]}(\{\exp r^{\mu}\}^{\frac{\lambda_{h_1}-2\epsilon}{\rho_{f_1}+\epsilon}})\})}{\log r} \quad from \ (3.3)$$

$$\ge limsup_{r\to\infty} \frac{\log M_g^{-1}(\exp\{\frac{1}{\rho_{f_2}+\epsilon}\log(r^{\mu}\frac{\lambda_{h_1}-2\epsilon}{\rho_{f_1}+\epsilon})\})}{\log r}$$

$$\ge limsup_{r\to\infty} \frac{\log M_g^{-1}(\exp\{\frac{\mu}{\rho_{f_2}+\epsilon}\log r + \frac{1}{\rho_{f_2}+\epsilon}\log\frac{\lambda_{h_1}-2\epsilon}{\rho_{f_1}+\epsilon})\})}{\log r}$$

$$\geq \limsup_{r\to\infty} \frac{\log M_g^{-1}(r^{\frac{\mu}{\rho_{f_2}+\epsilon}})+O(1)}{\log r}$$

$$\geq \limsup_{r\to\infty} \frac{1}{m}\frac{\log(\frac{r^{\frac{\mu}{\rho_{f_2}+\epsilon}}}{|a_m|(1+\epsilon)})+O(1)}{\log r} \quad from \ (3.7)$$

$$\geq \frac{1}{m}\frac{\mu}{\rho_{f_2}+\epsilon}$$

$$\geq \frac{1}{m}\frac{\rho_{h_2}}{\rho_{f_2}}, \ since \ \ \epsilon>0 \ is \ \ arbitrary.$$

We know from the definition of lower order for all large values of r

$$M_{f_1}^{-1}(r) < \exp\{\frac{1}{\lambda_{f_1}-\epsilon}\log^{[2]} r\}. \qquad (3.20)$$

Similarly, for all large values of r

$$M_{f_2}^{-1}(r) < \exp\{\frac{1}{\lambda_{f_2}-\epsilon}\log^{[2]} r\}. \qquad (3.21)$$

Also $T(r,h_1) < r^{\rho_{h_1}+\epsilon}$ *for all large values of* r.

Again by Lemma 2.2we get

$$T_{f_1\circ f_2\circ g}(r) \geq \frac{1}{3}\log M_{f_1}(\frac{1}{9}M_{f_2\circ g}(\frac{r}{4}))$$

$$\geq \frac{1}{3}\log M_{f_1}(\frac{1}{9}M_{f_2}(\frac{1}{10}M_g(\frac{r}{8})))$$

$$i.e, \quad T_{f_1\circ f_2\circ g}^{-1}(r) \leq 8M_g^{-1}(10M_{f_2}^{-1}(9M_{f_1}^{-1}(\exp 3r))). \qquad (3.22)$$

Also for a sequence of values of r tending to infinity we get from Lemma 2.8

$$T_{h_1\circ h_2}(r) \leq \{1+0(1)\}T_{h_1}(\exp(r)^{\mu}) \quad where \ \ \mu > \rho_{h_2}. \qquad (3.23)$$

Now

$$\rho_{f_1\circ f_2\circ g}(h_1\circ h_2) = \limsup_{r\to\infty} \frac{\log T_{f_1\circ f_2\circ g}^{-1}(T_{h_1\circ h_2}(r))}{\log r}$$

$$\leq \limsup_{r\to\infty} \frac{\log(8M_g^{-1}(10M_{f_2}^{-1}(9M_{f_1}^{-1}(\exp(3T_{h_1\circ h_2}(r))))))}{\log r} \quad from \ (3.22)$$

$$\leq \limsup_{r\to\infty} \frac{\log(8M_g^{-1}(10M_{f_2}^{-1}(9M_{f_1}^{-1}(\exp(3\{1+0(1)\}T_{h_1}(\exp r^{\mu}))))))}{\log r} \quad from \ (3.23)$$

$$\leq \limsup_{r\to\infty} \frac{\log(8M_g^{-1}(10M_{f_2}^{-1}(9\exp\{\frac{1}{\lambda_{f_1}-\epsilon}\log^{[2]}(\exp(3T_{h_1}(\exp r^{\mu})))\})))+O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\log(8M_g^{-1}(10M_{f_2}^{-1}(9\exp\{\frac{1}{\lambda_{f_1}-\epsilon}\log(3T_{h_1}(\exp r^{\mu}))\}))) + O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\log(8M_g^{-1}(10M_{f_2}^{-1}(9[3T_{h_1}(\exp r^{\mu})]^{\frac{1}{\lambda_{f_1}-\epsilon}}))) + O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\log(8M_g^{-1}(10M_{f_2}^{-1}(9[3(\exp r^{\mu})^{\rho_{h_1}+\epsilon}]^{\frac{1}{\lambda_{f_1}-\epsilon}}))) + O(1)}{\log r}\ \ from\ \ (3.15)$$

$$\leq limsup_{r\to\infty}\frac{\log(8M_g^{-1}(10\exp\{\frac{1}{\lambda_{f_2}-\epsilon}\log^{[2]}(9[3(\exp r^{\mu})^{\rho_{h_1}+\epsilon}]^{\frac{1}{\lambda_{f_1}-\epsilon}})\})) + O(1)}{\log r}\ \ from\ \ (3.21)$$

$$\leq limsup_{r\to\infty}\frac{\log(8M_g^{-1}(10\exp\{\frac{1}{\lambda_{f_2}-\epsilon}\log(r^{\mu}\frac{\rho_{h_1}+2\epsilon}{\lambda_{f_1}-\epsilon})\})) + O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\log(8M_g^{-1}(10\exp\{\frac{\mu}{\lambda_{f_2}-\epsilon}\log r+\frac{1}{\lambda_{f_2}-\epsilon}\log\frac{\rho_{h_1}+2\epsilon}{\lambda_{f_1}-\epsilon}\})) + O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\log(8M_g^{-1}(10\exp\{\log r^{\frac{\mu}{\lambda_{f_2}-\epsilon}}\})) + O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\log(8M_g^{-1}(10r^{\frac{\mu}{\lambda_{f_2}-\epsilon}})) + O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{\log(M_g^{-1}(10r^{\frac{\mu}{\lambda_{f_2}-\epsilon}})) + O(1)}{\log r}$$

$$\leq limsup_{r\to\infty}\frac{1}{m}\frac{\log(\frac{10r^{\frac{\mu}{\lambda_{f_2}-\epsilon}}}{|a_m|(1-\epsilon)}) + O(1)}{\log r}\ \ from\ \ (3.7)$$

$$\leq limsup_{r\to\infty}\frac{1}{m}\frac{(\log 10+\frac{\mu}{\lambda_{f_2}-\epsilon}\log r-\log|a_m|(1-\epsilon)) + O(1)}{\log r}$$

$$\leq\frac{1}{m}\frac{\mu}{\lambda_{f_2}-\epsilon}$$

$$\leq\frac{1}{m}\frac{\rho_{h_2}}{\lambda_{f_2}},\ since\ \epsilon>0\ \ is\ \ arbitrary.$$

Theorem 3.4 *Let f_1 and h_1 be two meromorphic functions of finite non-zero orders such that $\rho_{f_1} \neq 0$ and f_2 and h_2 be two entire functions of finite non-zero orders such that $\rho_{f_2} \neq 0$ and g be a polynomial of degree m. Then the relative order of $h_1 \circ h_2$ with respect to $f_1 \circ f_2 \circ g$ satisfies the inequality*

$$\rho_{f_1 \circ f_2 \circ g}(h_1 \circ h_2) \geq \frac{\lambda_{h_1}}{\rho_{f_1}} \quad and \quad \rho_{f_1 \circ f_2 \circ g}(h_1 \circ h_2) \leq \frac{\rho_{h_1}}{m\lambda_{f_2}}.$$

Proof: Let $g(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$ be a polynomial of degree m. We have from Lemma 2.5

$$T_{h_1 \circ h_2}(r) \geq T_{h_1}(\exp(r)^{\mu}). \tag{3.24}$$

Again for all large values of r we get from Lemma 2.7

$$\begin{aligned} T_{f_1 \circ f_2 \circ g}(r) &\leq T_{f_1}(M_{f_2 \circ g}(r)) \\ &\leq T_{f_1}(M_{f_2}(M_g(r))). \end{aligned} \tag{3.25}$$

Now by the definition of relative order we get

$$\rho_{f_1 \circ f_2 \circ g}(h_1 \circ h_2) = \limsup_{r \to \infty} \frac{\log T_{h_1 \circ h_2}(r)}{\log T_{f_1 \circ f_2 \circ g}(r)}$$

$$\geq \limsup_{r \to \infty} \frac{\log T_{h_1}(\exp r^{\mu})}{\log T_{f_1 \circ f_2 \circ g}(r)} \quad from \ (3.24)$$

$$\geq \limsup_{r \to \infty} \frac{\log T_{h_1}(\exp r^{\mu})}{\log T_{f_1}(M_{f_2}(M_g(r)))} \quad from \ (3.25)$$

$$\geq \limsup_{r \to \infty} \frac{\log T_{h_1}(\exp r^{\mu})}{\log T_{f_1}(M_{f_2}(|a_m|(1+\varepsilon)r^m))} \quad from \ (3.6)$$

$$\geq \limsup_{r \to \infty} \frac{\log T_{h_1}(\exp r^{\mu})}{\log T_{f_1}(\exp[|a_m|(1+\varepsilon)r^m]^{\rho_{f_2}+\varepsilon})}$$

$$\geq \limsup_{r \to \infty} \frac{\log T_{h_1}(\exp r^{\mu})}{(\rho_{f_1}+\varepsilon)[|a_m|(1+\varepsilon)r^m]^{\rho_{f_2}+\varepsilon}} \quad from \ (3.15)$$

$$\geq \limsup_{r \to \infty} \frac{(\lambda_{h_1}-\varepsilon)r^{\mu}}{(\rho_{f_1}+\varepsilon)[|a_m|(1+\varepsilon)r^m]^{\rho_{f_2}+\varepsilon}}$$

$$\geq \frac{\lambda_{h_1}}{\rho_{f_1}} \quad when \ \mu > m\rho_{f_2} \ \ and \ \ \varepsilon > 0 \ is \ arbitrary.$$

Also we have from Lemma 2.6

$$T_{h_1 \circ h_2}(r) \leq T_{h_1}(M_{h_2}(r)). \tag{3.26}$$

Also for a sequence of values of r tending to infinity we get from Lemma 2.4

$$T_{f_1\circ f_2\circ g}(r) \geq T_{f_2\circ g}(\exp r^{\mu}) \quad for \quad \mu < \rho_{f_2\circ g}$$

$$\geq \log M_{f_2\circ g}(\exp r^{\mu})$$

$$\geq \log M_{f_2}(\frac{1}{8}M_g(\frac{\exp r^{\mu}}{4})). \qquad (3.27)$$

Now by the definition of relative order we get

$$\rho_{f_1\circ f_2\circ g}(h_1\circ h_2) = \limsup_{r\to\infty}\frac{\log T_{h_1\circ h_2}(r)}{\log T_{f_1\circ f_2\circ g}(r)}$$

$$\leq \limsup_{r\to\infty}\frac{\log T_{h_1}(M_{h_2}(r))}{\log T_{f_1\circ f_2\circ g}(r)} \quad from \ (3.26)$$

$$\leq \limsup_{r\to\infty}\frac{\log T_{h_1}(M_{h_2}(r))}{\log^{[2]}(M_{f_2}(\frac{1}{8}(M_g(\frac{\exp r^{\mu}}{4}))))} \quad from \ (3.27)$$

$$\leq \limsup_{r\to\infty}\frac{\log T_{h_1}(M_{h_2}(r))}{\log^{[2]}(M_{f_2}(\frac{1}{8}|a_m|(1-\epsilon)[\frac{\exp r^{\mu}}{4}]^m))))} \quad from \ (3.6)$$

$$\leq \limsup_{r\to\infty}\frac{\log T_{h_1}(\exp\{r^{\rho_{h_2}+\epsilon}\})}{\log^{[2]}(M_{f_2}(\frac{1}{8}|a_m|(1-\epsilon)[\frac{\exp r^{\mu}}{4}]^m))}$$

$$\leq \limsup_{r\to\infty}\frac{(\rho_{h_1}+\epsilon)\log[\exp\{r^{\rho_{h_2}+\epsilon}\}]}{(\lambda_{f_2}-\epsilon)\log(\frac{1}{8}|a_m|(1-\epsilon)(\frac{\exp r^{\mu}}{4})^m)}$$

$$\leq \limsup_{r\to\infty}\frac{(\rho_{h_1}+\epsilon)\{r^{\rho_{h_2}+\epsilon}\}}{(\lambda_{f_2}-\epsilon)mr^{\mu}+O(1)}$$

$$\leq \frac{\rho_{h_1}}{m\lambda_{f_2}} \quad when \quad \mu < \rho_{h_2}.$$

Theorem 3.5 Let f_1 and h_1 be two meromorphic functions of finite non-zero order and f_2, h_2 and g be three entire functions such that $\rho_{f_2} > 0.$ Then

$$\rho_{h_1\circ h_2}(f_1\circ f_2\circ g) = \infty.$$

Proof: From the definition of relative order of meromorphic function with respect to another meromorphic function we have

$$\rho_{h_1\circ h_2}(f_1\circ f_2\circ g) = \limsup_{r\to\infty}\frac{\log T_{f_1\circ f_2\circ g}(r)}{\log T_{h_1\circ h_2}(r)}.$$

Also for a sequence of values of r tending to infinity

$M_g(r) > \exp\{r^{\rho_g - \varepsilon}\}$. (3.28)

Now

$$\rho_{h_1 \circ h_2}(f_1 \circ f_2 \circ g) \geq \limsup_{r\to\infty} \frac{\log^{[2]} M_{f_2}(\frac{1}{8} M_g(\frac{\exp r^{\mu}}{4}))}{\log T_{h_1 \circ h_2}(r)} \quad from \ (3.27)$$

$$\geq \limsup_{r\to\infty} \frac{\log^{[2]} M_{f_2}(\frac{1}{8} M_g(\frac{\exp r^{\mu}}{4}))}{\log T_{h_1}(M_{h_2}(r))} \quad from \ (3.26)$$

$$\geq \limsup_{r\to\infty} \frac{\log^{[2]} M_{f_2}(\frac{1}{8} \exp(\frac{\exp r^{\mu}}{4})^{\rho_g - \varepsilon})}{\log T_{h_1}(M_{h_2}(r))}$$

$$\geq \limsup_{r\to\infty} \frac{\log^{[2]} M_{f_2}(\frac{1}{8} \exp(\frac{\exp r^{\mu}}{4})^{\rho_g - \varepsilon})}{\log T_{h_1}(\exp(r^{\rho_{h_2} + \varepsilon}))}$$

$$\geq \limsup_{r\to\infty} \frac{(\rho_{f_2} - \varepsilon)(\frac{\exp r^{\mu}}{4})^{\rho_g - \varepsilon}}{(\rho_{h_1} + \varepsilon)(r^{\rho_{h_2} + \varepsilon})} \to \infty$$

Thus $\rho_{h_1 \circ h_2}(f_1 \circ f_2 \circ g) = \infty$.

This completes the proof.

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