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Successive differentiations of tangent, cotangent, secant, cosecant functions and related hyperbolic functions (A hypergeometric approach)

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Abstract: *In this paper, we obtain the expansions of $tan(x)$ and $cot(x)$ by taking logarithmic derivative of the infinite products for $cos(x)$ and $sin(x)$ respectively. Applying trigonometric identities, series rearrangement technique (for unilateral and bilateral infinite series) and suitable substitutions, we obtain the expansions of $cosec(x)$ and $sec(x)$. We also obtain the expansions, hypergeometric forms, convergence conditions and explicit formulas for m-th derivatives of $tan(ax+b)$, $cot(ax+b)$, $cosec(ax+b)$, $sec(ax+b)$ and related hyperbolic functions.*

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1. INTRODUCTION AND PRELIMINARIES

- **Pochhammer's symbol $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined by**

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & ;(\nu=0;\lambda\in\mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)...(\lambda+n-1) & ;(\nu=n\in\mathbb{N};\lambda\in\mathbb{C}\backslash\mathbb{Z}_0^-) \\ \frac{(-1)^n k!}{(k-n)!} & ;(\lambda=-k;\nu=n;n,k\in\mathbb{N}_0;0\le n\le k) \\ 0 & ;(\lambda=-k;\nu=n;n,k\in\mathbb{N}_0;n>k) \\ \frac{(-1)^n}{(1-\lambda)_n} & ;(\nu=-n;n\in\mathbb{N};\lambda\in\mathbb{C}\setminus\mathbb{Z} \end{cases} \tag{1.1}$$

where $\Gamma(\lambda)$ is the familiar Gamma function and it is obvious that $(0)_0 = 1$,
$\mathbb{N} = \{1, 2, ...\}$; $\mathbb{N}_0 : \mathbb{N} \cup \{0\}$; $\mathbb{Z}_0^- : \mathbb{Z}^- \cup \{0\} = \{0, -1, -2, ...\}$.
Here the symbols $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of natural numbers and integers respectively, as usual the symbols $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers respectively.

- **The generalized hypergeometric function ${}_pF_q$ [17,p.42,eq.(1)]**

$${}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,...,\alpha_p; \\ \\ \beta_1,\beta_2,...,\beta_q;\end{array} z\right] = {}_pF_q\left[\begin{array}{c}(\alpha_p); \\ \\ (\beta_q);\end{array} z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!}, \tag{1.2}$$

where $\alpha_i \in \mathbb{C},\ (i = 1, ..., p),\ \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ (j = 1, ..., q)$ and (α_p) represents the set of p numerator parameters given by $\alpha_1, \alpha_2, \ldots, \alpha_p$ and argument z take complex values

$$(\alpha_1)_r\ (\alpha_2)_r \ldots (\alpha_p)_r = \prod_{i=1}^{p} (\alpha_i)_r = \prod_{i=1}^{p} \frac{\Gamma(\alpha_i + r)}{\Gamma(\alpha_i)}, \tag{1.3}$$

with similar interpretation for others.

• **Convergence conditions for $_pF_q$ [17,p.43,eq.(5)]**

When $p, q \in \mathbb{N}_0;\ p \leqq q + 1$ and

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i,$$

(i) $p \leqq q$ and $|z| < \infty$ then $_pF_q$ is convergent,
(ii) $p = q + 1$ and $|z| < 1$ then $_pF_q$ is convergent,
(iii) $p = q + 1, |z| = 1$ and $\Re(\omega) > 0$ then $_pF_q$ is absolutely convergent,
(iv) $p = q + 1, |z| = 1, z \neq 1$ and $-1 < \Re(\omega) \leq 0$ then $_pF_q$ is conditionally convergent,
where $\Re$ denotes the real part.

• **If A and P are suitably adjusted real or complex numbers such that associated Pochhammer's symbols are well-defined, then**

$$A + Pn = A \frac{\left(\frac{A+P}{P}\right)_n}{\left(\frac{A}{P}\right)_n}; \quad n = \ 0,\ 1,\ 2,\ 3, ... \quad . \tag{1.4}$$

• **Some important identities related to unilateral infinite series**

$$\sum_{n=0}^{\infty} \Phi(n) = \sum_{n=0}^{\infty} \Phi(2n) + \sum_{n=0}^{\infty} \Phi(2n + 1), \tag{1.5}$$

$$\sum_{n=0}^{\infty} \Phi(n) = \sum_{n=0}^{\infty} \Phi(3n) + \sum_{n=0}^{\infty} \Phi(3n + 1) + \sum_{n=0}^{\infty} \Phi(3n + 2), \tag{1.6}$$

$$\sum_{n=0}^{\infty} \Phi(n) = \sum_{n=0}^{\infty} \Phi(4n) + \sum_{n=0}^{\infty} \Phi(4n + 1) + \sum_{n=0}^{\infty} \Phi(4n + 2) + \sum_{n=0}^{\infty} \Phi(4n + 3). \tag{1.7}$$

• **General form of the equations (1.5), (1.6) and (1.7) [17,p.218(Q.N.8)]**

$$\sum_{n=0}^{\infty} \Phi(n) = \sum_{j=0}^{m-1} \sum_{n=0}^{\infty} \Phi(mn + j), \tag{1.8}$$

where $m = 1, 2, 3, ...$

• **Some important identities related to bilateral infinite series**

$$\sum_{n=-\infty}^{\infty} \Phi(2n + 1) = \sum_{n=0}^{\infty} \Phi(-2n - 1) + \sum_{n=0}^{\infty} \Phi(2n + 1), \tag{1.9}$$

$$\sum_{n=-\infty}^{\infty} \Phi(2n) = \sum_{n=0}^{\infty} \Phi(-2n - 2) + \sum_{n=0}^{\infty} \Phi(2n), \tag{1.10}$$

$$\sum_{n=-\infty}^{\infty} \Phi(n) = \sum_{n=0}^{\infty} \Phi(-n - 1) + \sum_{n=0}^{\infty} \Phi(n), \tag{1.11}$$

$$\sum_{n=-\infty}^{\infty}\Phi(n)=\Phi(0)+\sum_{n=0}^{\infty}\Phi(n+1)+\sum_{n=0}^{\infty}\Phi(-n-1)=\sum_{n=-\infty}^{\infty}\Phi(-n). \tag{1.12}$$

• **Successive differentiation of reciprocal of a linear polynomial**

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}(ax+b)^{-1}=\frac{(a)^m(m)!(-1)^m}{(ax+b)^{m+1}}. \tag{1.13}$$

• **Successive differentiation of p-th degree polynomial**

$$\frac{\mathrm{d}^m(x)^p}{\mathrm{d}x^m}=\frac{(p)!}{(p-m)!}(x)^{p-m};\quad p\geq m, \tag{1.14}$$

$$\frac{\mathrm{d}^m(x)^p}{\mathrm{d}x^m}=0;\quad p<m. \tag{1.15}$$

• **The m-th derivatives of $\sin(ax+b)$ and $\cos(ax+b)$**

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\Big\{\sin(ax+b)\Big\}=(a)^m\sin\left(\frac{m\pi}{2}+ax+b\right), \tag{1.16}$$

$$=(a)^{m+1}x\cos\left(b+\frac{m\pi}{2}\right)\ {}_0F_1\left[\begin{array}{c}\overline{}\ ;\\ 3/2\ ;\end{array}-\frac{(ax)^2}{4}\right]+(a)^m\sin\left(b+\frac{m\pi}{2}\right)\ {}_0F_1\left[\begin{array}{c}\overline{}\ ;\\ 1/2\ ;\end{array}-\frac{(ax)^2}{4}\right]. \tag{1.17}$$

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\Big\{\cos(ax+b)\Big\}=(a)^m\cos\left(\frac{m\pi}{2}+ax+b\right), \tag{1.18}$$

$$=(a)^m\cos\left(b+\frac{m\pi}{2}\right)\ {}_0F_1\left[\begin{array}{c}\overline{}\ ;\\ 1/2\ ;\end{array}-\frac{(ax)^2}{4}\right]-(a)^{m+1}x\sin\left(b+\frac{m\pi}{2}\right)\ {}_0F_1\left[\begin{array}{c}\overline{}\ ;\\ 3/2\ ;\end{array}-\frac{(ax)^2}{4}\right]. \tag{1.19}$$

• **Logarithm of an infinite product**

$$\log_e\left\{\prod_{m=1}^{\infty}(b_m)\right\}=\sum_{m=1}^{\infty}\Big\{\log_e(b_m)\Big\}. \tag{1.20}$$

• **Relations between circular and hyperbolic functions**

$$\sin(ix)=(i)\sinh(x), \tag{1.21}$$

$$\cos(ix)=\cosh(x), \tag{1.22}$$

$$\tan(ix)=(i)\tanh(x), \tag{1.23}$$

$$\cot(ix)=(-i)\coth(x), \tag{1.24}$$

$$\sec(ix)=\mathrm{sech}(x), \tag{1.25}$$

$$\mathrm{cosec}(ix)=(-i)\mathrm{cosech}(x). \tag{1.26}$$

• **Trigonometric identity**

$$\mathrm{cosec}(x)=\cot\left(\frac{x}{2}\right)-\cot(x), \tag{1.27}$$

Motivated by the work of Adegoke and Layeni [1], Boyadzhiev [2], Daboul *et. al* [3], Hoffman [6], Lampret [7], Ma[10;11], Qi[12;13;14] and Qureshi *et. al*[15;16] and the equations (1.17) and (1.19), we obtain the successive differentiations, hypergeometric forms and convergence conditions of some trigonometric functions and related hyperbolic functions.

2. EXPANSIONS OF tan(x), cot(x), cosec(x), sec(x), sec^2(x), cosec^2(x) AND RELATED HYPERBOLIC FUNCTIONS

Any values of parameters and variables leading to the results which do not make sense are tacitly excluded.

$$\tan(x) = (8x)\sum_{n=0}^{\infty}\left[\frac{1}{(2n+1)^2\pi^2 - 4x^2}\right], \tag{2.1}$$

where $x \neq \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2}, ...$

$$\tanh(x) = (8x)\sum_{n=0}^{\infty}\left[\frac{1}{(2n+1)^2\pi^2 + 4x^2}\right], \tag{2.2}$$

where $x \neq \pm\frac{i\pi}{2},\ \pm\frac{3i\pi}{2},\ \pm\frac{5i\pi}{2}, ...$

$$\cot(x) = \frac{1}{x} + (2x)\sum_{n=0}^{\infty}\left[\frac{1}{x^2 - (n+1)^2\pi^2}\right], \tag{2.3}$$

where $x \neq 0,\ \pm\pi,\ \pm 2\pi,\ \pm 3\pi, ...$

$$\coth(x) = \frac{1}{x} + (2x)\sum_{n=0}^{\infty}\left[\frac{1}{x^2 + (n+1)^2\pi^2}\right], \tag{2.4}$$

where $x \neq 0,\ \pm i\pi,\ \pm 2i\pi,\ \pm 3i\pi, ...$

$$\sec(x) = (4\pi)\sum_{n=0}^{\infty}\left[\frac{(-1)^n(2n+1)}{(2n+1)^2\pi^2 - 4x^2}\right], \tag{2.5}$$

where $x \neq \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2}, ...$

$$\text{sech}(x) = (4\pi)\sum_{n=0}^{\infty}\left[\frac{(-1)^n(2n+1)}{(2n+1)^2\pi^2 + 4x^2}\right], \tag{2.6}$$

where $x \neq \pm\frac{i\pi}{2},\ \pm\frac{3i\pi}{2},\ \pm\frac{5i\pi}{2}, ...$

$$\text{cosec}(x) = \frac{1}{x} + (2x)\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(n+1)^2\pi^2 - x^2}\right], \tag{2.7}$$

where $x \neq 0,\ \pm\pi,\ \pm 2\pi,\ \pm 3\pi, ...$

$$\text{cosech}(x) = \frac{1}{x} - (2x)\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(n+1)^2\pi^2 + x^2}\right], \tag{2.8}$$

where $x \neq 0,\ \pm i\pi,\ \pm 2i\pi,\ \pm 3i\pi, ...$

$$\sec^2(x) = (4)\sum_{n=-\infty}^{\infty}\left[\frac{1}{\{(2n+1)\pi + 2x\}^2}\right], \tag{2.9}$$

where $x \neq \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2}, ...$

$$\text{sech}^2(x) = (4)\sum_{n=-\infty}^{\infty}\left[\frac{1}{\{(2n+1)\pi + 2ix\}^2}\right], \tag{2.10}$$

where $x \neq \pm\frac{i\pi}{2},\ \pm\frac{3i\pi}{2},\ \pm\frac{5i\pi}{2}, ...$

$$\text{cosec}^2(x) = \sum_{n=-\infty}^{+\infty}\left[\frac{1}{(x-n\pi)^2}\right], \tag{2.11}$$

where $x \neq 0,\ \pm\pi,\ \pm 2\pi,\ \pm 3\pi, ...$

$$\text{cosech}^2(x) = (-1)\sum_{n=-\infty}^{+\infty}\left[\frac{1}{(ix-n\pi)^2}\right], \tag{2.12}$$

where $x \neq 0,\ \pm i\pi,\ \pm 2i\pi,\ \pm 3i\pi, ...$

3. SYSTEMATIC PROOFS FOR THE EXPANSIONS OF $\tan(x)$, $\cot(x)$, $\text{cosec}(x)$, $\sec(x)$, $\sec^2(x)$ AND $\text{cosec}^2(x)$ BY SERIES REARRANGEMENT TECHNIQUE

• **Proof for the expansion of $\tan(x)$:**
We know that the infinite product of $\cos(x)$[4,p.345;5,p.348;8,p.441;9,p.151] in powers of x, is given by

$$\cos(x) = \prod_{n=1}^{\infty}\left[1-\frac{4x^2}{(2n-1)^2\pi^2}\right], \tag{3.1}$$

Now using logarithmic property (1.20), we get

$$\log_e\{\cos(x)\} = \sum_{n=1}^{\infty}\left[\log_e\left\{1-\frac{4x^2}{(2n-1)^2\pi^2}\right\}\right]. \tag{3.2}$$

Differentiate both sides of the equation (3.2) with respect to x, after simplification, we get

$$\tan(x) = (8x)\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)^2\pi^2-4x^2}\right], \tag{3.3}$$

where $x \neq \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2}, ...$
Now replacing n by $n+1$ in the equation (3.3) , we get the equation (2.1) .

• **Proof for the expansion of $\cot(x)$:**
We know that the infinite product of $\sin(x)$[4,p.345;5,p.348;8,p.439;9,p.149] in powers of x, is given by

$$\sin(x) = (x)\prod_{n=1}^{\infty}\left[1-\frac{x^2}{n^2\pi^2}\right]. \tag{3.4}$$

Using logarithmic property (1.20), we get

$$\log_e\{\sin(x)\} = \log_e(x) + \sum_{n=1}^{\infty}\left[\log_e\left\{1-\frac{x^2}{n^2\pi^2}\right\}\right], \tag{3.5}$$

Differentiate both sides of the equation (3.5) with respect to x, after simplification, we get

$$\cot(x) = \frac{1}{x} + (2x)\sum_{n=1}^{\infty}\left[\frac{1}{x^2-n^2\pi^2}\right], \tag{3.6}$$

where $x \neq 0,\ \pm\pi,\ \pm 2\pi,\ \pm 3\pi, ...$
Now replacing n by $n+1$ in the equation (3.6), we get the equation (2.3).

• **Proof for the expansion of cosec(x):**
Replacing x by $\frac{x}{2}$ in the equation (3.6), we get

$$\cot\left(\frac{x}{2}\right) = \frac{2}{x} + \sum_{n=1}^{\infty}\left[\frac{4x}{x^2 - 4n^2\pi^2}\right]. \tag{3.7}$$

Subtracting the equation (3.6) from the equation (3.7) and applying the trigonometric identity (1.27), we get

$$\operatorname{cosec}(x) = \frac{2}{x} + \sum_{n=1}^{\infty}\left[\frac{4x}{x^2 - 4n^2\pi^2}\right] - \frac{1}{x} - \sum_{n=1}^{\infty}\left[\frac{2x}{x^2 - n^2\pi^2}\right]. \tag{3.8}$$

Replacing n by $n+1$ in the equation (3.8), we get

$$\operatorname{cosec}(x) = \frac{1}{x} + \sum_{n=0}^{\infty}\left[\frac{4x}{x^2 - 4(n+1)^2\pi^2}\right] - \sum_{n=0}^{\infty}\left[\frac{2x}{x^2 - (n+1)^2\pi^2}\right]. \tag{3.9}$$

Now applying the even-odd infinite series identity (1.5), we get

$$\operatorname{cosec}(x) = \frac{1}{x} + \sum_{n=0}^{\infty}\left[\frac{4x}{x^2 - (2n+2)^2\pi^2}\right] - \sum_{n=0}^{\infty}\left[\frac{2x}{x^2 - (2n+1)^2\pi^2}\right] - \sum_{n=0}^{\infty}\left[\frac{2x}{x^2 - (2n+2)^2\pi^2}\right]. \tag{3.10}$$

$$\operatorname{cosec}(x) = \frac{1}{x} + \sum_{n=0}^{\infty}\left[\frac{2x}{x^2 - (2n+2)^2\pi^2}\right] - \sum_{n=0}^{\infty}\left[\frac{2x}{x^2 - (2n+1)^2\pi^2}\right]. \tag{3.11}$$

Equation (3.11) can also be written as

$$\operatorname{cosec}(x) = \frac{1}{x} + \sum_{n=0}^{\infty}\left[\frac{(2x)(-1)^{2n+2}}{x^2 - (2n+2)^2\pi^2}\right] + \sum_{n=0}^{\infty}\left[\frac{(2x)(-1)^{2n+1}}{x^2 - (2n+1)^2\pi^2}\right]. \tag{3.12}$$

Again using the even-odd infinite series identity (1.5) in the equation (3.12), we get the equation (2.7).

• **Proof for the expansion of sec(x):**
Equation (2.7) can also be written in the form of partial fractions

$$\operatorname{cosec}(x) = \frac{1}{x} + \sum_{n=0}^{\infty}(-1)^{n+1}\left[\frac{1}{x + (n+1)\pi} + \frac{1}{x - (n+1)\pi}\right]. \tag{3.13}$$

Replacing x by $\left(\frac{\pi}{2} + x\right)$ in the equation (3.13), we get

$$\sec(x) = \frac{1}{\left(x + \frac{\pi}{2}\right)} + \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}}{\left(x + \frac{\pi}{2}\right) + (n+1)\pi}\right] + \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}}{\left(x + \frac{\pi}{2}\right) - (n+1)\pi}\right]. \tag{3.14}$$

Put $n = 0$ in second infinite series of right hand side of the equation (3.14), we have

$$\sec(x) = \left[\frac{1}{\left(x + \frac{\pi}{2}\right)} - \frac{1}{\left(x - \frac{\pi}{2}\right)}\right] + \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}}{\left(x + \frac{\pi}{2}\right) + (n+1)\pi}\right] + \sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{\left(x + \frac{\pi}{2}\right) - (n+1)\pi}\right]. \tag{3.15}$$

Now replacing n by $n+1$ in second infinite series of right hand side of the equation (3.15), we have

$$\sec(x) = \left[\frac{1}{\left(x + \frac{\pi}{2}\right)} - \frac{1}{\left(x - \frac{\pi}{2}\right)}\right] - \sum_{n=0}^{\infty}\left[\frac{(-1)^{n}}{\left(x + \frac{\pi}{2}\right) + (n+1)\pi}\right] + \sum_{n=0}^{\infty}\left[\frac{(-1)^{n}}{\left(x + \frac{\pi}{2}\right) - (n+2)\pi}\right]. \tag{3.16}$$

After simplification, we get

$$\sec(x) = \left[\frac{1}{\left(x + \frac{\pi}{2}\right)} - \frac{1}{\left(x - \frac{\pi}{2}\right)}\right] - \sum_{n=0}^{\infty}(-1)^{n}\left[\frac{1}{x + \left(n + \frac{3}{2}\right)\pi} - \frac{1}{x - \left(n + \frac{3}{2}\right)\pi}\right]. \tag{3.17}$$

Now replacing n by $n-1$ and applying the series rearrangement technique in the equation (3.17), we have

$$\sec(x) = \left[\frac{1}{\left(x+\frac{\pi}{2}\right)} - \frac{1}{\left(x-\frac{\pi}{2}\right)}\right] + \sum_{n=1}^{\infty}(-1)^{n-1}\left[-\frac{1}{x+\left(n+\frac{1}{2}\right)\pi} + \frac{1}{x-\left(n+\frac{1}{2}\right)\pi}\right]. \tag{3.18}$$

$$\sec(x) = \sum_{n=0}^{\infty}(-1)^{n-1}\left[\frac{1}{x-\left(n+\frac{1}{2}\right)\pi} - \frac{1}{x+\left(n+\frac{1}{2}\right)\pi}\right]. \tag{3.19}$$

After simplification, we get the equation (2.5).

• **Proof for the expansion of $\sec^2(x)$:**

Equation (3.3) can also be written in the form of partial fractions

$$\tan(x) = (2)\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)\pi-2x} - \frac{1}{(2n-1)\pi+2x}\right]. \tag{3.20}$$

Differentiate both sides of the equation (3.20) with respect to x, we get

$$\sec^2(x) = (4)\sum_{n=1}^{\infty}\left[\frac{1}{\{-(2n-1)\pi+2x\}^2} + \frac{1}{\{(2n-1)\pi+2x\}^2}\right]. \tag{3.21}$$

Replacing n by $n+1$ in the equation (3.21), we get

$$\sec^2(x) = (4)\sum_{n=0}^{\infty}\left[\frac{1}{\{(2n+1)\pi-2x\}^2} + \frac{1}{\{(2n+1)\pi+2x\}^2}\right]. \tag{3.22}$$

Applying the identity (1.9) for bilateral series in the equation (3.22), we get the equation (2.9).

•**Proof for the expansion of $\mathrm{cosec}^2(x)$:**

Equation (3.6) can also be written in the form of partial fractions

$$\cot(x) = \frac{1}{x} + \sum_{n=1}^{\infty}\left[\frac{1}{x+n\pi} + \frac{1}{x-n\pi}\right]. \tag{3.23}$$

Differentiate both sides of the equation (3.23) with respect to x, we get

$$\mathrm{cosec}^2(x) = \frac{1}{x^2} + \sum_{n=1}^{\infty}\left[\frac{1}{(x+n\pi)^2} + \frac{1}{(x-n\pi)^2}\right]. \tag{3.24}$$

Replacing n by $n+1$ in the equation (3.24) we have

$$\mathrm{cosec}^2(x) = \frac{1}{x^2} + \sum_{n=0}^{\infty}\left[\frac{1}{\{x+(n+1)\pi\}^2} + \frac{1}{\{x-(n+1)\pi\}^2}\right]. \tag{3.25}$$

Applying the bilateral series identity in the equation (3.25), we get the equation (2.11).
Now using the relations (1.21) to (1.26) in the expansions (2.1), (2.3), (2.5), (2.7), (2.9) and (2.11), we get the remaining expansions (2.2), (2.4), (2.6), (2.8), (2.10) and (2.12) respectively.

4. EXPANSIONS FOR THE m-th DERIVATIVES OF $\tan(ax+b)$, $\cot(ax+b)$, $\mathrm{cosec}(ax+b)$, $\sec(ax+b)$ AND RELATED HYPERBOLIC FUNCTIONS

Any values of parameters and variables leading to the results which do not make sense are tacitly excluded.

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\{\tan(ax+b)\} = (2)\sum_{n=0}^{\infty}\left[\frac{(2a)^m\, m!}{\{-2ax+(2n+1)\pi-2b\}^{m+1}} - \frac{(2a)^m(-1)^m m!}{\{2ax+(2n+1)\pi+2b\}^{m+1}}\right], \tag{4.1}$$

where $ax+b \neq \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2},...$

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\{\tanh(ax+b)\} = (2)\sum_{n=0}^{\infty}\left[\frac{(2a)^m(-1)^m m!}{\{2ax+(2n+1)i\pi+2b\}^{m+1}} + \frac{(2a)^m(-1)^m m!}{\{2ax-(2n+1)i\pi+2b\}^{m+1}}\right], \tag{4.2}$$

where $ax+b \neq \pm\frac{i\pi}{2},\ \pm\frac{3i\pi}{2},\ \pm\frac{5i\pi}{2},...$

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\{\cot(ax+b)\} = \frac{(a)^m(-1)^m m!}{(ax+b)^{m+1}} + \sum_{n=0}^{\infty}\left[\frac{(a)^m(-1)^m m!}{\{ax+b+(n+1)\pi\}^{m+1}} + \frac{(a)^m(-1)^m m!}{\{ax+b-(n+1)\pi\}^{m+1}}\right], \tag{4.3}$$

where $ax+b \neq 0,\ \pm\pi,\ \pm2\pi,\ \pm3\pi,...$

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\{\coth(ax+b)\} = \frac{(a)^m(-1)^m m!}{(ax+b)^{m+1}} + \sum_{n=0}^{\infty}\left[\frac{(a)^m(-1)^m m!}{\{ax+b-(n+1)i\pi\}^{m+1}} + \frac{(a)^m(-1)^m m!}{\{ax+b+(n+1)i\pi\}^{m+1}}\right], \tag{4.4}$$

where $ax+b \neq 0,\ \pm i\pi,\ \pm2i\pi,\ \pm3i\pi,...$

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\{\mathrm{cosec}(ax+b)\} = \frac{(a)^m(-1)^m m!}{(ax+b)^{m+1}} - \sum_{n=0}^{\infty}\left[\frac{(-1)^{m+n}(a)^m m!}{\{ax+b+(n+1)\pi\}^{m+1}} + \frac{(-1)^{m+n}(a)^m m!}{\{ax+b-(n+1)\pi\}^{m+1}}\right], \tag{4.5}$$

where $ax+b \neq 0,\ \pm\pi,\ \pm2\pi,\ \pm3\pi,...$

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\{\mathrm{cosech}(ax+b)\} = \frac{(a)^m(-1)^m m!}{(ax+b)^{m+1}} - \sum_{n=0}^{\infty}\left[\frac{(-1)^{m+n}(a)^m m!}{\{ax+b-(n+1)i\pi\}^{m+1}} + \frac{(-1)^{m+n}(a)^m m!}{\{ax+b+(n+1)i\pi\}^{m+1}}\right], \tag{4.6}$$

where $ax+b \neq 0,\ \pm i\pi,\ \pm2i\pi,\ \pm3i\pi,...$

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\{\sec(ax+b)\} = (2)\sum_{n=0}^{\infty}\left[\frac{(-1)^{m+n}(2a)^m m!}{\{2ax+2b+(2n+1)\,\pi\}^{m+1}} - \frac{(-1)^{m+n}(2a)^m m!}{\{2ax+2b-(2n+1)\,\pi\}^{m+1}}\right], \tag{4.7}$$

where $ax+b \neq \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2},...$

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\{\mathrm{sech}(ax+b)\} = (2)\sum_{n=0}^{\infty}\left[\frac{(-1)^{m+n}(2ia)^m m!}{\{2iax+2ib+(2n+1)\,\pi\}^{m+1}} - \frac{(-1)^{m+n}(2ia)^m m!}{\{2iax+2ib-(2n+1)\,\pi\}^{m+1}}\right], \tag{4.8}$$

where $ax+b \neq \pm\frac{i\pi}{2},\ \pm\frac{3i\pi}{2},\ \pm\frac{5i\pi}{2},...$

5. PROOF FOR THE m-th DERIVATIVE OF $\tan(ax+b)$

Replacing n by $n+1$ and x by $ax+b$ in the equation (3.20), we have

$$\tan(ax+b) = (2)\sum_{n=0}^{\infty}\left[\frac{1}{(-2a)x+(2n+1)\pi-2b} - \frac{1}{(2a)x+(2n+1)\pi+2b}\right]. \tag{5.1}$$

Differentiate both sides of the equation (5.1), m-times with respect to x and using differential formula (1.13), we get the equation (4.1).
Applying same process in the equations (2.3), (2.5), (2.7), we can find the m-th derivatives for $\cot(ax+b)$, $\sec(ax+b)$, $\mathrm{cosec}(ax+b)$ and related hyperbolic functions.

6. HYPERGEOMETRIC FORMS FOR THE m-th DERIVATIVE OF $\tan(ax+b)$, $\cot(ax+b)$, $\mathrm{cosec}(ax+b)$, $\sec(ax+b)$ AND RELATED HYPERBOLIC FUNCTIONS

Any values of parameters and variables leading to the results which do not make sense are tacitly excluded.

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\{\tan(ax+b)\} = \left\{\frac{(2)(2a)^m m!}{(-2ax+\pi-2b)^{m+1}}\right\}{}_{m+2}F_{m+1}\left[\begin{array}{c} 1, \overbrace{\frac{-2ax+\pi-2b}{2\pi},\ldots,\frac{-2ax+\pi-2b}{2\pi}}^{m+1}; \\ \underbrace{\frac{-2ax+3\pi-2b}{2\pi},\ldots,\frac{-2ax+3\pi-2b}{2\pi}}_{m+1}; \end{array} 1\right] - $$
$$-\left\{\frac{(2)(2a)^m(-1)^m m!}{(2ax+\pi+2b)^{m+1}}\right\}{}_{m+2}F_{m+1}\left[\begin{array}{c} 1, \overbrace{\frac{2ax+\pi+2b}{2\pi},\ldots,\frac{2ax+\pi+2b}{2\pi}}^{m+1}; \\ \underbrace{\frac{2ax+3\pi+2b}{2\pi},\ldots,\frac{2ax+3\pi+2b}{2\pi}}_{m+1}; \end{array} 1\right], \quad (6.1)$$

where $ax+b \neq \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2},\ldots$

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\{\tanh(ax+b)\} = \left\{\frac{(2)(2a)^m(-1)^m m!}{(2ax+i\pi+2b)^{m+1}}\right\}{}_{m+2}F_{m+1}\left[\begin{array}{c} 1, \overbrace{\frac{-2iax+\pi-2ib}{2\pi},\ldots,\frac{-2iax+\pi-2ib}{2\pi}}^{m+1}; \\ \underbrace{\frac{-2iax+3\pi-2ib}{2\pi},\ldots,\frac{-2iax+3\pi-2ib}{2\pi}}_{m+1}; \end{array} 1\right] + $$
$$+\left\{\frac{(2)(2a)^m(-1)^m m!}{(2ax-i\pi+2b)^{m+1}}\right\}{}_{m+2}F_{m+1}\left[\begin{array}{c} 1, \overbrace{\frac{2iax+\pi+2ib}{2\pi},\ldots,\frac{2iax+\pi+2ib}{2\pi}}^{m+1}; \\ \underbrace{\frac{2iax+3\pi+2ib}{2\pi},\ldots,\frac{2iax+3\pi+2ib}{2\pi}}_{m+1}; \end{array} 1\right], \quad (6.2)$$

where $ax+b \neq \pm\frac{i\pi}{2},\ \pm\frac{3i\pi}{2},\ \pm\frac{5i\pi}{2},\ldots$

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\{\cot(ax+b)\} = \left\{\frac{(a)^m(-1)^m m!}{(ax+b)^{m+1}}\right\} + \\ + \left\{\frac{(a)^m(-1)^m m!}{(ax+b+\pi)^{m+1}}\right\} {}_{m+2}F_{m+1}\left[\begin{array}{c} 1,\ \overbrace{\frac{ax+b+\pi}{\pi},...,\frac{ax+b+\pi}{\pi}}^{m+1}; \\ \underbrace{\frac{ax+b+2\pi}{\pi},...,\frac{ax+b+2\pi}{\pi}}_{m+1}; \end{array} 1\right] + \\ + \left\{\frac{(a)^m(-1)^m m!}{(ax+b-\pi)^{m+1}}\right\} {}_{m+2}F_{m+1}\left[\begin{array}{c} 1,\ \overbrace{\frac{-ax-b+\pi}{\pi},...,\frac{-ax-b+\pi}{\pi}}^{m+1}; \\ \underbrace{\frac{-ax-b+2\pi}{\pi},...,\frac{-ax-b+2\pi}{\pi}}_{m+1}; \end{array} 1\right], \tag{6.3}$$

where $ax+b \neq 0,\ \pm\pi,\ \pm 2\pi,\ \pm 3\pi, ...$

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\{\coth(ax+b)\} = \left\{\frac{(a)^m(-1)^m m!}{(ax+b)^{m+1}}\right\} + \\ + \left\{\frac{(a)^m(-1)^m m!}{(ax+b-i\pi)^{m+1}}\right\} {}_{m+2}F_{m+1}\left[\begin{array}{c} 1,\ \overbrace{\frac{iax+ib+\pi}{\pi},...,\frac{iax+ib+\pi}{\pi}}^{m+1}; \\ \underbrace{\frac{iax+ib+2\pi}{\pi},...,\frac{iax+ib+2\pi}{\pi}}_{m+1}; \end{array} 1\right] + \\ + \left\{\frac{(a)^m(-1)^m m!}{(ax+b+i\pi)^{m+1}}\right\} {}_{m+2}F_{m+1}\left[\begin{array}{c} 1,\ \overbrace{\frac{-iax-ib+\pi}{\pi},...,\frac{-iax-ib+\pi}{\pi}}^{m+1}; \\ \underbrace{\frac{-iax-ib+2\pi}{\pi},...,\frac{-iax-ib+2\pi}{\pi}}_{m+1}; \end{array} 1\right], \tag{6.4}$$

where $ax+b \neq 0,\ \pm i\pi,\ \pm 2i\pi,\ \pm 3i\pi, ...$

$$\frac{d^m}{dx^m}\{\text{cosec}(ax+b)\} = \left\{\frac{(a)^m(-1)^m m!}{(ax+b)^{m+1}}\right\} - \left\{\frac{(a)^m(-1)^m m!}{(ax+b+\pi)^{m+1}}\right\}{}_{m+2}F_{m+1}\left[\begin{matrix} 1, \overbrace{\frac{ax+b+\pi}{\pi},...,\frac{ax+b+\pi}{\pi}}^{\text{m+1}}; \\ \underbrace{\frac{ax+b+2\pi}{\pi},...,\frac{ax+b+2\pi}{\pi}}_{\text{m+1}}; \end{matrix} -1\right] - \left\{\frac{(a)^m(-1)^m m!}{(ax+b-\pi)^{m+1}}\right\}{}_{m+2}F_{m+1}\left[\begin{matrix} 1, \overbrace{\frac{-ax-b+\pi}{\pi},...,\frac{-ax-b+\pi}{\pi}}^{\text{m+1}}; \\ \underbrace{\frac{-ax-b+2\pi}{\pi},...,\frac{-ax-b+2\pi}{\pi}}_{\text{m+1}}; \end{matrix} -1\right], \tag{6.5}$$

where $ax+b \neq 0,\ \pm\pi,\ \pm 2\pi,\ \pm 3\pi, ...$

$$\frac{d^m}{dx^m}\{\text{cosech}(ax+b)\} = \left\{\frac{(a)^m(-1)^m m!}{(ax+b)^{m+1}}\right\} - \left\{\frac{(a)^m(-1)^m m!}{(ax+b-i\pi)^{m+1}}\right\}{}_{m+2}F_{m+1}\left[\begin{matrix} 1, \overbrace{\frac{iax+ib+\pi}{\pi},...,\frac{iax+ib+\pi}{\pi}}^{\text{m+1}}; \\ \underbrace{\frac{iax+ib+2\pi}{\pi},...,\frac{iax+ib+2\pi}{\pi}}_{\text{m+1}}; \end{matrix} -1\right] - \left\{\frac{(a)^m(-1)^m m!}{(ax+b+i\pi)^{m+1}}\right\}{}_{m+2}F_{m+1}\left[\begin{matrix} 1, \overbrace{\frac{-iax-ib+\pi}{\pi},...,\frac{-iax-ib+\pi}{\pi}}^{\text{m+1}}; \\ \underbrace{\frac{-iax-ib+2\pi}{\pi},...,\frac{-iax-ib+2\pi}{\pi}}_{\text{m+1}}; \end{matrix} -1\right], \tag{6.6}$$

where $ax+b \neq 0,\ \pm i\pi,\ \pm 2i\pi,\ \pm 3i\pi, ...$

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\{\sec(ax+b)\} = \left\{\frac{(2)^{m+1}(a)^m(-1)^m m!}{(2ax+2b+\pi)^{m+1}}\right\}{}_{m+2}F_{m+1}\left[\begin{array}{c} 1,\overbrace{\frac{2ax+2b+\pi}{2\pi},...,\frac{2ax+2b+\pi}{2\pi}}^{\mathrm{m+1}}; \\ \underbrace{\frac{2ax+2b+3\pi}{2\pi},...,\frac{2ax+2b+3\pi}{2\pi}}_{\mathrm{m+1}}; \end{array} -1\right] -$$
$$-\left\{\frac{(2)^{m+1}(a)^m(-1)^m m!}{(2ax+2b-\pi)^{m+1}}\right\}{}_{m+2}F_{m+1}\left[\begin{array}{c} 1,\overbrace{\frac{-2ax-2b+\pi}{2\pi},...,\frac{-2ax-2b+\pi}{2\pi}}^{\mathrm{m+1}}; \\ \underbrace{\frac{-2ax-2b+3\pi}{2\pi},...,\frac{-2ax-2b+3\pi}{2\pi}}_{\mathrm{m+1}}; \end{array} -1\right], \tag{6.7}$$

where $ax+b \neq \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2},...$

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\{\mathrm{sech}(ax+b)\} = \left\{\frac{(2)^{m+1}(ai)^m(-1)^m m!}{(2aix+2ib+\pi)^{m+1}}\right\}{}_{m+2}F_{m+1}\left[\begin{array}{c} 1,\overbrace{\frac{2aix+2ib+\pi}{2\pi},...,\frac{2aix+2ib+\pi}{2\pi}}^{\mathrm{m+1}}; \\ \underbrace{\frac{2aix+2ib+3\pi}{2\pi},...,\frac{2aix+2ib+3\pi}{2\pi}}_{\mathrm{m+1}}; \end{array} -1\right] -$$
$$-\left\{\frac{(2)^{m+1}(ai)^m(-1)^m m!}{(2aix+2ib-\pi)^{m+1}}\right\}{}_{m+2}F_{m+1}\left[\begin{array}{c} 1,\overbrace{\frac{-2aix-2ib+\pi}{2\pi},...,\frac{-2aix-2ib+\pi}{2\pi}}^{\mathrm{m+1}}; \\ \underbrace{\frac{-2aix-2ib+3\pi}{2\pi},...,\frac{-2aix-2ib+3\pi}{2\pi}}_{\mathrm{m+1}}; \end{array} -1\right] \tag{6.8}$$

where $ax+b \neq \pm\frac{i\pi}{2},\ \pm\frac{3i\pi}{2},\ \pm\frac{5i\pi}{2},...$

7. PROOF OF THE HYPERGEOMETRIC FORM FOR THE m-th DERIVATIVE OF $\tan(ax+b)$

Equation (4.1) can also be written as,

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\{\tan(ax+b)\} = (2)\sum_{n=0}^{\infty}\left[\frac{(2a)^m m!}{\{-2ax+\pi-2b+2n\pi\}^{m+1}}\right] - (2)\sum_{n=0}^{\infty}\left[\frac{(2a)^m(-1)^m m!}{\{2ax+\pi+2b+2n\pi\}^{m+1}}\right]. \tag{7.1}$$

On using Pochhammer symbol identity (1.4) for the following linear polynomials, we have

$$(-2ax+\pi-2b)+2\pi(n) = \left\{\frac{(-2ax+\pi-2b)\left(\frac{-2ax+3\pi-2b}{2\pi}\right)_n}{\left(\frac{-2ax+\pi-2b}{2\pi}\right)_n}\right\}, \tag{7.2}$$

and

$$(2ax+\pi+2b)+2\pi(n)=\left\{\frac{(2ax+\pi+2b)\left(\frac{2ax+3\pi+2b}{2\pi}\right)_n}{\left(\frac{2ax+\pi+2b}{2\pi}\right)_n}\right\}. \tag{7.3}$$

Now using the equations (7.2) and (7.3) in the equation (7.1), we get

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left\{\tan(ax+b)\right\}=\frac{(2)(2a)^m m!}{(-2ax+\pi-2b)^{m+1}}\sum_{n=0}^{\infty}\left[\frac{(1)_n\left\{\left(\frac{-2ax+\pi-2b}{2\pi}\right)_n\right\}^{m+1}(1)^n}{\left\{\left(\frac{-2ax+3\pi-2b}{2\pi}\right)_n\right\}^{m+1}n!}\right]-$$

$$-\frac{(2)(2a)^m(-1)^m m!}{(2ax+\pi+2b)^{m+1}}\sum_{n=0}^{\infty}\left[\frac{(1)_n\left\{\left(\frac{2ax+\pi+2b}{2\pi}\right)_n\right\}^{m+1}(1)^n}{\left\{\left(\frac{2ax+3\pi+2b}{2\pi}\right)_n\right\}^{m+1}n!}\right]. \tag{7.4}$$

Now using the definition of generalized hypergeometric function (1.2) in the equation (7.4), we get the equation (6.1).
Applying same process in the equations (4.3), (4.5), (4.7), we can find the hypergeometric forms for the m-th derivatives of $\cot(ax+b)$, $\operatorname{cosec}(ax+b)$, $\sec(ax+b)$ and related hyperbolic functions.

8. CONVERGENCE CONDITIONS FOR THE HYPERGEOMETRIC FORMS OF THE m-th DERIVATIVES

- **When argument is unity**

and $\Re$ (sum of denominator's parameters−sum of numerator's parameters) $>\ 0$
then ${}_{m+2}F_{m+1}(1)$ will be convergent.
From the equation (6.1), we have

$$\Re\ \left[\left(\frac{-2ax+3\pi-2b}{2\pi}\right)(m+1)-\left\{1+\left(\frac{-2ax+\pi-2b}{2\pi}\right)(m+1)\right\}\right]=m;\quad m\geq 1, \tag{8.1}$$

and

$$\Re\ \left[\left(\frac{2ax+3\pi+2b}{2\pi}\right)(m+1)-\left\{1+\left(\frac{2ax+\pi+2b}{2\pi}\right)(m+1)\right\}\right]=m;\quad m\geq 1. \tag{8.2}$$

Therefore $\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left\{\tan(ax+b)\right\}$ is convergent, where $m=1,\,2,\,3...$.

- **When argument is -1**

and $\Re$ (sum of denominator's parameters−sum of numerator's parameters) $>\ -1$,
then ${}_{m+2}F_{m+1}(-1)$ will be convergent.
Similarly, we can prove that, our remaining results (6.2) to (6.8) are convergent.

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