$\beta^{\#}$ - I- Continuous Multifunctions

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Abstract. *In this paper, we study a new type of continuity of multifunctions by using $\beta^{\#}$ $-I-$ open sets. Several characterizations and some properties of these multifunctions are obtained. Relationships with other kinds of $I-$ continuity of multifunctions are investigated.*

Keywords: β −open set, $\beta^{\#}$ $-I-$open set, Multifunctions, $\beta^{\#}$ $-I-$continuous multifunctions,

I. INTRODUCTION

In [1] C. Berge introduced the theory of multifunctions. A multifunction is a set-valued function. The concept of multifunctions has applications in functional analysis and fixed point theory. The notion of ideal in topological space was first introduced by Kuratowski[2] and Vaidyanathswamy [3]. In 1990, D. Jankovi´c and T. R. Hamlett [4] introduced the notion of I−open sets in topological spaces. In 1992, Abd El-Monsef et al. [5] further investigated I−open sets and I−continuous functions. Dontchev [6] introduced the concept of pre-I−open sets and obtained a decomposition of I−continuity. Hatır and Noiri [7] introduced the notion of semi−I−open sets and α−I−open sets to obtain decomposition of continuity. In [8] the notion of weakly semi-I-open (Which we called $\beta^{\#}$ -I- open) sets was introduced by Hatir and Jafari. Akdag [9] introduced and stady the I-continuous multifunction. In [10], the concepts of upper (lower) α−I−continuous multifunctions on ideal topological spaces are studied. The notion of semi−I−continuous multifunctions was studies in [11]. In [12] Adiya introduced and study the concepts of upper (lower) β −I−continuous multifunctions on ideal topological spaces. In the present paper, we introduce and study the concepts of $\beta^{\#}$ −I−continuous multifunctions on ideal topological spaces. Some characterizations and properties are obtained. Also, we investigate its relationships with other types of I−continuities of multifunctions.

II. PRELIMINARIES

An ideal is a nonempty collection I of subsets of X satisfying the following two conditions: (1) $A \in I$ and $B \subset A$ implies $B \in I$ (2) If $A \in I$ and $B \in I$, then $A \cup B \in I$.
An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I). An operator $(.)^*$: P(X) → P(X), is called the local function[6] of I on X with respect to τ and I is defined as follows: For $A \subset \mathbf{X}$, $A^*(I,\tau)$ = {x ∈ X : G(1) $A \in I$ and $B \subset A$ implies $B \in I$ (2) If $A \in I$ and $B \in I$, then $A \cup B \in I$.
An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I). An operator $(.)^*$: P(X) → P(X), is called the local function[6] of I on X with respect to τ and I is defined as follows: For $A \subset \mathbf{X}$, $A^*(I,\tau)$ = {x ∈ X : G ∩ A ∉ I for every G ∈ τ (x)} where τ (x) = {G ∈ τ : x ∈ G}[3]. Moreover, cl^*(A) = A $\bigcup$ A^* defines a Kuratowski closure operator for a topology τ^* (I, τ) which is finer than τ. For any ideal space (X, τ, I), the collection $\{U \setminus G : U \in \tau, G \in I\}$ is a base for τ^* (I, τ).

A multifunction of a set X into Y will be denoted by F: X ↦ Y. For a multifunction F, the upper and lower inverse set of a set B of Y will be denoted by $F^+(B)$ and $F^-(B)$ respectively that is $F^+(B) = \{x \in X : F(x) \subseteq B\}$ and $F^-(B) = \{x \in X : F(x) \bigcap B \neq \phi\}$. A multifunction F : X ↦ Y is said to be upper semi continuous (briefly u. s. c) at a point x ∈ X if for each open set V in Y with F(x) ⊆ V , there exists an open set U containing x such that F(U) ⊆ V ; lower semi continuous (briefly l. s. c.) at a point x ∈ X if for each open set V in Y with F(x)∩V ≠ ϕ , there exists an open set U containing x such that F(z) ∩ V ≠ ϕ for every z ∈ U [13]. Throughout this paper, A^c denote the complement of A. Spaces X and Y mean topological spaces and int(A) and Cl(A) denote the interior and closure of A respectively.

III. $\beta^{\#}$ – I–Continuity of Multifunctions

Definition 3. 1 A subset A of an ideal topological space (X, τ, I) is said to be

(1) I–open if A $\subseteq$ int (A^*)[5]

(2) α – I–open if A ⊂ Int (Cl* (int (A)))[7]

(3) β – I–open if A $\subseteq$ (Cl (int (Cl* (A)))[16]

(4) pre – I–open if A $\subseteq$ Int (Cl* (A))[6]

(5) semi – I–open if A $\subseteq$ Cl* (int (A))[7]

(6) pre–open if A $\subseteq$ Int (Cl (A))[14]

(7) semi–open if A $\subseteq$ Cl (int (A))[15]

(8) $\beta^{\#}$ -I- open (or weakly semi-I- open relative to [8]) set if A $\subseteq$ Cl*(Int(Cl(A)))

Using [12] and [8, Remarks 2.1 and 2.2], we have the following daigram,

open $\Rightarrow \alpha - I -$ open $\Rightarrow$ semi $- I -$ open $\Rightarrow$ semi – open $\Rightarrow \beta - I -$ open

$I -$ open $\Rightarrow$ pre $- I -$ open $\Rightarrow$ pre – open $\Rightarrow \beta^{\#}$ -I- open $\Rightarrow \beta$ - open

We introduce the following definition,

Definition 3.2 A multifunction F :(X, τ, I) $\mapsto$ (Y, σ) is said to be upper (resp. lower) $\beta^{\#}$ –I–continuous iff for each x ∈ X and each open set V in Y with F(x) ⊂ V (resp. F(x) ∩ V ≠ ϕ), there exists $\beta^{\#}$ – I–open set U containing x such that F(U) = ∪{F(u) : u ∈ U} ⊂ V (resp. if u ∈ U, then F(u) ∩ V ≠ ϕ). We say that F is $\beta^{\#}$ -I- continuous if it is upper and lower $\beta^{\#}$ -I- continuous.

Now we introduce the following characterizations,

Theorem 3.3 Let F: (X, τ, I) $\mapsto$ (Y, σ) be a multifunction, then the following statements are equivalent:

(1) F is upper (resp. lower) $\beta^{\#}$ – I– continuous.

(2) For each x ∈ X and each open set V in Y with x ∈ F$^+$(V) (resp. x ∈ F^- (V)) ,there exists a $\beta^{\#}$ – I–open set U containing x such that U⊂ F$^+$(V) (resp. U⊂ F^- (V)).

(3) For every open set V in Y, F$^+$ (V) (resp. F$^-$ (V)) is a $\beta^{\#}$ – I–open set in X.

(4) For every closed set V in Y, F$^-$ (V) (resp. F^+ (V)) is a $\beta^{\#}$ – I–closed set in X.

(5) int*(Cl (int (F^- (V)))) ⊂ F^- (Cl(V)) (resp. int*(Cl (int (F$^+$(V)))) ⊂ F$^+$(Cl (V)) for any subset V of Y.

(6) F(int*(Cl (int(U)))) ⊂ (Cl (F (U)), for each subset U of X.

Proof. (1) $\Rightarrow$(2): Let x ∈ X and V be any open set in Y with x ∈ F$^+$ (V) (resp. x ∈ F^- (V)). Then F (x) ⊂ V (resp.F(x) ∩ V ≠ ϕ). Since F is upper (lower) $\beta^{\#}$ – I–continuous, there exists a $\beta^{\#}$ – I–open set U ontaining x such that F(U) ⊂V (resp. if u ∈ U, then F(u) ∩ V ≠ ϕ). Thus U⊂ F$^+$ (V) (resp. U⊂ F^- (V)).

(2) $\Rightarrow$(3): Let V be any open set in Y and let x ∈ F$^+$ (V) (resp. x ∈ (V)). Then by (2), there exists a $\beta^{\#}$ –I–open set U_x containing x such that F(U_x) ⊂V (resp. U_x ⊂ F^- (V)). Since the union of $\beta^{\#}$ – I–open sets is a $\beta^{\#}$ –I–open [8], F$^+$(V) = $\bigcup$ U_x (resp. F^- (V) = $\bigcup$ U_x) is a $\beta^{\#}$ – I–open set in X.

(3) $\Rightarrow$ (4): Let V be a closed set in Y. Hence Y – V is an open set in Y. Then by (3), F$^+$ (Y – V) = X – F^- (V) (resp. F^- (Y – V) = X – F$^+$(V)) is a $\beta^{\#}$ – I–open set in X. So F$^-$(V) (resp. $F^+(V)$) is a $\beta^{\#}$ – I–closed set in X.

(4)$\Rightarrow$ (5): Let V⊂ Y any subset of Y. Since Cl (V) is closed set in Y. By (4), (F^- (Cl (V)) (resp. F$^+$(Cl (V)) is $\beta^{\#}$ – I–closed set in X. Thus (F^- (Cl (V)))^c ⊂ Cl* (int (Cl(F^- (Cl(V)))^c)) = (int* (Cl (int (F^- (V))))^c

(resp. $(F^{+}(Cl(V)))^{c} \subset Cl^{*}(int(Cl(F^{+}(Cl(V)))^{c} = (int^{*}(Cl(int(F^{+}(V))))^{c}$. Hence $int^{*}(Cl(int(F^{-}(V))) \subset F^{-}(Cl(V))$ (resp. $int^{*}(Cl(int(F^{+}(Cl(V))))) \subset F^{+}(Cl(V)$).

(5) $\Rightarrow$ (6): Let U be any subset of X. By (5), we have $int^{*}(Cl(int(U))) \subset int^{*}(Cl(int(F^{-}(F(U))))) \subset (F^{-}Cl(F(U)))$ (resp. $int^{*}(Cl(int(U))) \subset int^{*}(Cl(int(F^{+}(F(U))))) \subset F^{+}(Cl(F(U))))$. Hence $F(int^{*}(Cl(int(U)))) \subset Cl(F(U))$.

(6) $\Rightarrow$ (1) Let V be any open subset of Y. Then by (6), $F(int^{*}(Cl(int(F^{-}(V^{c}))))) \subset Cl(F(F^{-}(V^{c}))) \subset Cl(V^{c})$ (resp. $F(int^{*}(Cl(int(F^{+}(V^{c}))))) \subset Cl(F(F+(V^{c}))) \subset Cl(V^{c})$). So $int^{*}(Cl(int(F^{-}(V^{c})))) \subset Cl(F^{-}(V^{c})) \subset (F^{+}(V))^{c}$ (resp. $int(Cl(int(F^{+}(V^{c})))) \subset Cl(F^{+}(V^{c})) \subset (F^{-}(V))^{c}$. Hence $F^{+}(V) \subset Cl^{*}(int(Cl(F^{+}(V)))$ (resp. $F^{-}(V) \subset Cl^{*}(int(Cl(F^{-}(V))))$. Hence $F^{+}(V)$(resp. $F^{-}(V)$) is $\beta^{\#}$−I−open. Thus, F is upper (lower) $\beta^{\#}$−I−continuous.

Definition 3.4 A multifunction F: X $\mapsto$ Y is called

(1) upper (resp. lower) pre-continuous if for every open set V in Y, $F^{+}(V)$(resp. $F^{-}(V)$) is a pre-open set in X.[17]

(2) upper (resp. lower) I-continuous if for every open set V in Y , F+(V) (resp. $F^{-}(V)$) is an I−open set in X. [9]

(3) upper (resp. lower) pre-I−continuous if for every open set V in Y , $F^{+}(V)$(resp. $F^{-}(V)$) is a pre- I−open set in X.[11]

(4) upper (resp. lower) semi-I−continuous if for every open set V in Y , $F^{+}(V)$(resp. $F^{-}(V)$) is a semi-I−open set in X.[11]

(5) upper (resp. lower) semi continuous if for every open set V in Y , $F^{+}(V)$ (resp. $F^{-}(V)$) is a semi open set in X. [14].

(6) upper (resp. lower) α−I−continuous iff for each $x \in X$ and each open set V in Y with $F(x) \subset V$ (resp. $F(x) \cap V \neq \phi$), there exists α −I−open set U containing x such that $F(U) = \cup\{F(u) : u \in U\} \subset V$ (resp. if $u \in U$, then $F(u) \cap V \neq \phi$).[10]

(7) upper (resp. lower) β −I−continuous[12] iff for each $x \in X$ and each open set V in Y with $F(x) \subset V$ (resp. $F(x) \cap V \neq \phi$), there exists β− I−open set U containing x such that $F(U) = \cup\{F(u) : u \in U\} \subset V$ (resp. if $u \in U$, then $F(u) \cap V \neq \phi$).

A multifunction F is called pre- continuous (resp., I- continuous, pre- I- continuous, semi-I- continuous, semi-continuous, α−I−continuous and β −I−continuous) if it is both upper and lower continuous for each of the above kinds.

From [12] and Definition 3.2, we have the following Diagram,

$$
\begin{array}{ccccccc}
\text{Continuity} & \Rightarrow & \alpha - I - \text{continuity} & \Rightarrow & \text{semi} - I - \text{continuity} & \overset{2}{\Rightarrow} & \beta^{\#} - \text{I–continuity} \\
 & & \Downarrow & & \Downarrow_{1} & & \Downarrow \\
I - \text{continuity} & \Rightarrow & \text{pre} - I - \text{continuity} & \Rightarrow & \beta - I - \text{continuity} & \Rightarrow & \beta\text{ - continuity}
\end{array}
$$

(Give an example to show that the converse of implication (2) is not true in general).

Remark 3.5 The converses of the above implications is not true in general, for the converse of implication (1) see [1] and for other converses see [7],[11].

Theorem 3.6 Let F: (X, τ, I) $\mapsto$ (Y, σ, J) be a multifunction and $\{U_{\lambda} : \lambda \in \Delta\}$ be an open cover of X. If the restriction functions $F_{|U_{\lambda}}$ is upper $\beta^{\#}$ −I−continuous for each for each $\lambda \in \Delta$, then F is upper $\beta^{\#}$ − I−continuous.

Proof. Let V be any open subset of Y. Since $F_{|U_{\lambda}}$ is upper $\beta^{\#}$ −I−continuous for each $\lambda \in \Delta$, hence

${F_{|U_\lambda}}^{+}(V) = U_\lambda \cap F^{+}(V)$ is $\beta^{\#}-$ I−open set. Then $\bigcup_{\lambda\in\Delta} (U_\lambda \bigcap F^{+}(V)) = \bigcup_{\lambda\in\Delta} (U_\lambda) \bigcap F^{+}(V) =$ $X \bigcap F^{+}(V) = F^{+}(V)$ is β − I−open set. Therefore F is upper $\beta^{\#}-$ I− continuous.

Theorem 3.7 If F: (X, τ, I) ⟼ (Y, σ) is upper $\beta^{\#}-$ I−continuous (resp. lower $\beta^{\#}$ −I−continuous) and F(X) is a subspace of Y, then F: X ⟼ F(X) is upper $\beta^{\#}-$ I−continuous (lower $\beta^{\#}-$ I−continuous).

Proof. Since F: (X, τ, I) ⟼ (Y, σ) is upper $\beta^{\#}$−I−continuous (lower $\beta^{\#}$ −I−continuous), so for every open subset V of Y , $F^{+}(V \cap F(X)) = F^{+}(V) \cap F^{+}(F(X)) = F^{+}(V)$(resp. $F^{-}(V \cap F(X)) = F^{-}(V) \cap F^{-}(F(X)) = F^{-}(V)$) is $\beta^{\#}$ −I−open. Hence F: X ⟼ F(X) is upper $\beta^{\#}$ − I−continuous (resp. lower $\beta^{\#}$ − I−continuous).

IV. Some Applications

In this section we provide some applications on $\beta^{\#}$ − I−continuous multifunctions.

Definition 4.1 [18] An ideal topological space (X, τ, I) is said to be I−compact if for every I−open cover $\{\{W_\lambda : \lambda \in \Delta\}$, there exists a finite subset Δ_0 of Δ such that $(X - \cup \{W_\lambda : \lambda \in \Delta_0\}) \in I$.

We introduce the following definition,

Definition 4.2 An ideal topological space (X, τ, I) is said to be $\beta^{\#}$−I−compact if for every $\beta^{\#}-$ I−open cover $\{W_\lambda : \lambda \in \Delta\}$ of X, there exists a finite subset Δ_0 of Δ such that $(X - \cup \{W_\lambda : \lambda \in \Delta_0\}) \in I$.

Lemma 4.3. [9] For any surjective multifunction F: (X, τ, I) ⟼ (Y, σ), F(I) is an ideal on Y.

Theorem 4.4. Let (X, τ, I) is $\beta^{\#}-$ I− compact space and F :(X, τ, I) ⟼ (Y, σ) is upper $\beta^{\#}-$ I−continuous surjection. Then (Y, σ) is F(I)−compact.

Proof. Let F: X ⟼ Y be a upper $\beta^{\#}$−I−continuous surjection and $\{V_\lambda : \lambda \in \Delta\}$ be an open cover of Y. Then $\{F^{+}(V_\lambda) : \lambda \in \Delta\}$ is a $\beta^{\#}-$ I−open cover of X. Since X is $\beta^{\#}$ −I−compact, there exists a finite subset Δ_0 of Δ such that $(X - \cup \{F^{+}(V_\lambda) : \lambda \in \Delta_0\}) \in I$. Therefore by lemma 4.3, $F(X - \cup \{F^{+}(V_\lambda) : \lambda \in \Delta_0\}) = (Y - \cup \{V_\lambda : \lambda \in \Delta_0\}) \in F(I)$. Hence (Y, σ, F (I)) is F(I)−compact.

Definition 4.5 An ideal topological space (X, τ, I) is called $\beta^{\#}$ − I – Hausdorff if for each two distinct points x ≠ y there exists disjoint $\beta^{\#}-$ I−open sets U and V containing x and y respectively. Then we say that x and y are $\beta^{\#}-$ I−separated.

Theorem 4.6 Let F: (X, τ, I) ⟼ (Y, σ, J) be upper $\beta^{\#}$ − I – continuous multifunction such that F(x) is closed for each $x \in X$. If Y is normal space then X is $\beta^{\#}$ −I − Hausdorff where $F(x) \cap F(y) = \phi$ for each distinct x, y $\in X$.

Proof. Let x, y ∈ X be distinct. Then $F(x) \cap F(y) = \phi$. Since Y is normal space then there exist distinct open sets U and V containing F (x) and F (y) respectively. Thus $F^{+}(U)$ and $F^{+}(V)$ are disjoint $\beta^{\#}$ −I−open sets containing x and y respectively. Then X is $\beta^{\#}-$ I – Hausdorff.

Definition 4.7 A multifunction F :(X, τ, I) ⟼ (Y, σ, J) is said to be

(a) upper $\beta^{\#}$ −I−irresolute if $F^{+}(V)$ is $\beta^{\#}$−I−open in X for each $\beta^{\#}$ −I−open set V in Y.

(b) lower $\beta^{\#}$−I−irresolute if $F^{-}(V)$ is $\beta^{\#}$−I−open in X for each $\beta^{\#}$−I−open set V in Y.

(c) $\beta^{\#}$−I−irresolute if F is upper $\beta^{\#}$−I−irresolute and lower $\beta^{\#}$−I−irresolute.

Theorem 4.8 (1) If F is upper $\beta^{\#}-$ I−irresolute multifunction then F is upper $\beta^{\#}-$ I−continuous multifunction.

(2) If F is lower $\beta^{\#}$ −I −irresolute multifunction then F is lower $\beta^{\#}$ −I− continuous multifunction.

Proof. Obvious, since any open set is $\beta^{\#}-$ I−open set.

Theorem 4.9 Let F :(X, τ , I) $\mapsto$ (Y, σ, J) be upper $\beta^{\#}$ −I−irresolute multifunction and G : (Y, σ, J) $\mapsto$ (Z, μ) be upper $\beta^{\#}-$ I−continuous multifunction then $(G \circ F)$ is upper $\beta^{\#}$ − I−continuous multifunction.

Proof. Let V be any open set in Z. Since G is upper $\beta^{\#}-$ I−continuous. Then $G^{+}(V)$ is $\beta^{\#}-$ J−open in Y. Since F is upper $\beta^{\#}-$ I−irresolute, then $F^{+}(G^{+}(V)) = ((G \circ F)^{+}(V)$ is $\beta^{\#}-$ I−open in X. Thus $G \circ F$ is upper $\beta^{\#}$ −I−continuous.

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