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Results on Graceful Labeling

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Abstract

In this paper we define smoothness of graceful labeling and we have discussed smoothness of C_n, $K_{2,n}$ and P_n. Last Theorem shows that any length of path is smooth graceful graph.

Key words : labeling, smooth graceful labeling, bipartite graph.

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1 Introduction :

Let $G = (V, E)$ be a simple, undirected graph of size (p, q) i.e. $|V| = p$, $|E| = q$. For all standard terminology and notations we follow Harary (Harary 1972). We will give brief summery of definitions which are used in this paper.

Definition-1.1**:** If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.

Definition-1.2**:** A function f is called *graceful labeling* of a graph $G = (V, E)$ if $f : V \longrightarrow \{0, 1, \ldots, q\}$ is injective and the induce function $f^\star : E \longrightarrow \{1, 2, \ldots, q\}$ defined as $f^\star(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u, v) \in E$. A graph G is called *graceful graph* if it admits a graceful labeling.

Definition-1.3**:** A bipartite graceful graph G with graceful labeling f is said to be *smooth graceful graph* if it admits an injective map $g : V \longrightarrow \{0, 1, \ldots, \lfloor\frac{q-1}{2}\rfloor, \lfloor\frac{q+1}{2}\rfloor + l, \lfloor\frac{q+3}{2}\rfloor + l, \ldots, q + l\}$ such that its induce edge labeling map $g^\star : E \longrightarrow \{1 + l, 2 + l, \ldots, q + l\}$ defined as $g^\star(e) = |g(u) - g(v)|$, for every edge $e = (u, v) \in E$, for any $l \in N$ is a bijection.

Smooth graceful graph will help to produce new disconnected as well as connected graceful graphs.

2 Main Results :

Theorem-2.1 **:** A cycle C_n, $(n \equiv 0 \pmod 4)$ is a smooth graceful graph.

Proof : Let $v_1, v_2, \ldots, v_n$ be vertices of C_n. We know that C_n $(n \equiv 0 \pmod 4)$ is a bipartite graph and $f : V(C_n) \longrightarrow \{0, 1, \ldots, n\}$ defined by

$$\begin{aligned} f(v_i) &= \tfrac{i-1}{2}, && \forall\ i = 1, 3, \ldots, n-1. \\ &= q - (\tfrac{i-2}{2}), && \forall\ i = 2, 4, \ldots, \tfrac{n}{2} \\ &= q - (\tfrac{i}{2}), && \forall\ i = \tfrac{n}{2} + 2, \tfrac{n}{2} + 4, \ldots, n \end{aligned}$$

is graceful labeling for C_n, when $n \equiv 0 \pmod 4$.

Now we define $g : V(C_n) \longrightarrow \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor, \lfloor \frac{n+1}{2} \rfloor + l, \lfloor \frac{n+3}{2} \rfloor + l, \ldots, n+l\}$ such that its induce edge labeling map $g^\star : E(C_n) \longrightarrow \{1+l, 2+l, \ldots, q+l\}$ defined by

$$\begin{aligned} g(u) &= f(u),\ \forall\ u \in \{v_1, v_3, \ldots, v_{n-1}\} \\ &= f(u) + l, \quad \forall\ u \in \{v_2, v_4, \ldots, v_n\} \text{ and} \end{aligned}$$

$g^\star(e) = |g(u) - g(v)|$, for every edge $e = (u, v) \in E$.

Since for any $k \in \{1, 2, \ldots, n\}$, $e_k = (v_k, v_{k+1}) \in E$, by taking $v_{n+1} = v_1$,

$$\begin{aligned} g^\star(e_k) &= |g(v_k - g(v_{k+1})| \\ &= |f(v_k - f(v_{k+1})| + l \\ &= f^\star(e_k) + l, \text{ so } g^\star(E) = \{1+l, 2+l, \ldots, n+l\}. \end{aligned}$$

Therefore $g^\star$ is a bijective map. Hence C_n, $(n \equiv 0 \pmod 4)$ is a smooth graceful graph.

Theorem-2.2 **:** A complete bipartite graph $K_{2,n}$ is a smooth graceful graph.

Proof : Let $u_1, u_2, v_1, v_2, \ldots, v_n$ be vertices of $K_{2,n}$.

We know that $f : V(K_{2,n}) \longrightarrow \{0, 1, \ldots, 2n\}$ defined by

$f(v_i) = i - 1, \quad \forall\ i = 1, 2, \ldots, n.$

$f(u_i) = in, \quad \forall\ i = 1, 2$ is a graceful labeling $K_{2,n}$.

Now we define $g : V(K_{2,n}) \longrightarrow \{0, 1, \ldots, n-1, n+l, n+1+l, \ldots, n+l\}$ and its induce edge labeling map $g^\star : E(K_{2,n}) \longrightarrow \{1+l, 2+l, \ldots, n+l\}$ defined by

$$\begin{aligned} g(u) &= f(u) + l,\ \forall\ u \in \{u_1, u_2\} \\ &= f(v),\ \forall\ v \in \{v_1, v_2, \ldots, v_n\} \text{ and } g^\star(e) = |g(u) - g(v)|,\ \forall\ e = (u, v) \in E. \end{aligned}$$

Since for any $e = (u_i, v_j) \in E$ $(i = 1, 2, 1 \le j \le n)$,

$$g^\star(e) = |g(u_i) - g(v_j)|$$

$$= f(u_i) - f(v_j)| + l$$
$$= f^{\star}(e) + l.$$

So $g^{\star}(E) = \{1 + l, 2 + l, \ldots, 2n + l\}$. Therefore $g^{\star}$ is a bijection.

Hence $K_{2,n}$ is a smooth graceful graph.

Theorem−2.3 **:** A path P_n of length $n-1$ is a smooth graceful graph, $\forall\ n \in N$.

Proof : Let $v_1, v_2, \ldots, v_n$ be vertices of P_n. We know that P_n is a bipartite and graceful graph with graceful labeling $f : V(P_n) \longrightarrow \{0, 1, \ldots, n-1\}$ defined by

$$\begin{aligned} f(v_i) &= \tfrac{i-2}{2}, && \text{if } i \equiv 0 \pmod 2 \\ &= n - (\tfrac{i+1}{2}), && \text{if } i \equiv 1 \pmod 2,\ \forall\ i = 1, 2, \ldots, n. \end{aligned}$$

Now we define $g : V(P_n) \longrightarrow \{0, 1, \ldots, \lfloor \frac{n-2}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + l, \lfloor \frac{n+2}{2} \rfloor + l, \ldots, n-1+l\}$ and its induce edge labeling map $g^{\star} : E(P_n) \longrightarrow \{1 + l, 2 + l, \ldots, n - 1 + l\}$ defined by

$$\begin{aligned} g(v) &= f(v) + l, && \text{if } i \equiv 1 \pmod 2 \\ &= f(v), && \text{if } i \equiv 0 \pmod 2,\ \forall\ i = 1, 2, \ldots, n \end{aligned}$$

and $g^{\star}(e) = |g(u) - g(v)|$, for every $e = (u, v) \in E$.

Since for any $e = (v_i, v_{i+1}) \in E$ $(1 \leq i \leq n-1)$,

$$\begin{aligned} g^{\star}(e) &= |g(v_i) - g(v_{i+1})| \\ &= |f(v_i) - f(v_{i+1})| + l \\ &= f^{\star}(e) + l, \text{ we shall have } g^{\star}(E) = \{1 + l, 2 + l, \ldots, n - 1 + l\}. \end{aligned}$$

Therefore $g^{\star}$ is a bijection and so P_n of length $n-1$ is a smooth graceful, $\forall\ n \in N$.

3 Concluding Remarks :

We have introduced a new graph labeling is called smooth graceful labeling. We have proved that C_n $n \equiv 0 \pmod 4$, $K_{2,n}$, P_n are smooth graceful graphs. Using these we have got graceful labeling for two new families of graphs. Present work contribute five new results. The labeling pattern is demonstrated by illustrations.

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