Index of labeling

M M Jariya

Government Science College

Abstract

In this paper we have proved that the index of cordiality for K_n is atmost 4, when n can be expressed as sum of square of two integers and also it is atmost 4 for different conditions of d_i, where $d_i = e_f(1) - e_f(0)$ for some binary vertex labeling function f on K_n.

Key words : Complete graph, binary vertex labeling, index of cordiality for a graph.

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1. Introduction

Let G be a simple, undirected finite graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges. We follows Harary [4] for all basic terminology and standard notations. In this paper we have used following definition.

Definition-1.1 **:** A function $f : V(G) \longrightarrow \{0, 1\}$ is called *a binary vertex labeling* of a graph G and $f(v)$ is called *label of the vertex* v of G under f. The induced function $f^* : E(G) \longrightarrow \{0, 1\}$ is defined as $f^*(e) = f(u) + f(v) \pmod 2$, for every edge $e = (u, v) \in E(G)$. Let $v_f(0)$, $v_f(1)$ be number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$, $e_f(1)$ be number of edges of G having labels 0 and 1 respectively under f^*. A binary vertex labeling f of a graph G is called *a cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph which admits a cordial labeling is called *a cordial graph*.

Definition-1.2 **:** Let G be a graph and $G^{(1)}, G^{(2)}, \ldots, G^{(n)}$, $n \geq 2$ be n copies of G. Let $v \in V(G)$. The graph obtained by joining vertex v of $G^{(i)}$ with same vertex of $G^{(i+1)}$ by an edge, $\forall\ i = 1, 2, \ldots, n-1$ and v of $G^{(n)}$ with the same vertex of $G^{(1)}$ by an edge is called *cycle of* G. It is denoted by $C(n \cdot G)$. If we replace G by $C(n \cdot G)$, such graph becomes $C(n \cdot C(n \cdot G))$, we denote it by $C^2(n \cdot G)$. In general for any $t \geq 2$, $C^t(n \cdot G) = C(n \cdot C^{t-1}(n \cdot G))$. Therefore $C(n \cdot K_1) = C_n$.

Definition-1.3 **:** Let G be a connected graph. If union of n copies of G ($\bigcup_{i=1}^{n} G$) is cordial but $\bigcup_{i=1}^{l} G$ do not have cordial labeling for every $l < n$, the *index of cordiality* for G is n.

1.4 **Discussion on cordiality of** K_n **:** Kaneria and Jariya proved that The index of cordiality for $K_n (n \leq 105)$ is at the most 6. Also they have computed table-1 and table-2 which contain different type of vertex labels to find required conditions $|v_f(1) - v_f(0)| \leq 1$, $|e_f(1) - e_f(0)| \leq 1$ in union of specified copies of K_n. Last column of *table*$-1, 2$ shows maximum index of cordiality for K_n.

2. Main Results

Theorem-2.1 The index of cordiality for K_n is at the most 4, When n is sum of squares of two integers.

Proof : We assume $n = i^2 + j^2$, for some $i, j \in N$ and $i \geq j$. Here we prove that union of four copies of K_n is cordial. For this, it is enough to show that $d_t + d_l = 0$, for some $t, l \in N$ and $1 \leq t, l \leq max\{i, j\} + 1$. Here the following cases are to be considered :

Case$-$**I :** $i = j$. In this case n is even.

Moreover $d_1 = \frac{i^2+j^2}{2} = i^2$, $d_2 = i^2 - 2$, $d_3 = i^2 - 8$, $\ldots$, $d_k = i^2 - 2k + 4k - 2$.

If $d_1 + d_k = 0$, $d_k = i^2 - 2k^2 + 4k - 2 = -i^2$

$\Rightarrow 2i^2 = 2k^2 - 4k + 2$

$\Rightarrow i^2 = (k-1)^2$

$\Rightarrow k = i + 1$. In this case, we get $d_1 + d_{i+1} = 0$.

Case$-$**II :** $i = j + 1$. In this case, n is odd.

Moreover, $d_1 = \frac{1}{2}(i^2 + j^2 - 1) = i(i-1)$, $d_2 = i(i-1) - 4$, $d_3 = i(i-1) - 10$, $\ldots$, $d_k = i(i-1) - 2k(k-1)$.

If $d_1 + d_k = 0$, $i(i-1) - 2k(k-1) = -i(i-1)$

$\Rightarrow k = i$. So, we get $d_1 + d_i = 0$.

Case$-$**III :** $i = j + 2$. Here, n is even and

$d_1 = \frac{1}{2}(i^2 + (i-2)^2) = i^2 - 2i + 2$, $d_2 = i^2 - 2i$, $\ldots$, $d_k = i^2 - 2i + 2 - 2k^2 + 4k - 2$.

If $d_2 + d_k = 0$, $i^2 - 2i - 2k^2 + 4k = -i^2 + 2i$

$\Rightarrow k = i$ and so, we get $d_2 + d_i = 0$.

Case$-$**IV :** $i = j + 3$. Here, n is odd and

$d_1 = i^2 - 3i + 4$, $d_2 = i^2 - 3i$, $\ldots$, $d_k = i^2 - 3i - 4 - 2k(k-1)$.

If $d_2 + d_k = 0$, $i^2 - 3i + 4 - 2k(k-1) = -i^2 + 3i$

$\Rightarrow i^2 - 3i + 2 = k(k-1)$

$\Rightarrow (i-1)(i-2) = k(k-1)$

$\Rightarrow k = i - 1$ and so, we get $d_2 + d_{i-1} = 0$.

General Case : Take $i = j + s$ i.e. $s = i - j$.

Subcase$-$**I :** s is even.

In this case, either i, j both are even or both are odd and so $n = i^2 + j^2$ is even.

We prove that $d_t + d_l$ and $t = i - \frac{s}{2} - 1$, $l = \frac{s}{2} + 1$.

Since n is even, $d_1 = \frac{1}{2}(i^2 + j^2)$, $d_2 = \frac{1}{2}(i^2 + j^2) - 2$, $\ldots$,$d_k = \frac{1}{2}(i^2 + j^2) - 2k^2 + 4k - 2$.

$$\begin{aligned}
\Rightarrow d_t &= \tfrac{1}{2}(i^2 + j^2) - 2(i - \tfrac{s}{2} - 1)^2 - 4(i - \tfrac{s}{2} - 1) - 2 \\
&= \tfrac{1}{2}(i^2 + j^2) - 2(i^2 + 1 + \tfrac{s^2}{4} - is - 2i + s) - 4i + 2s + 2 \\
&= \tfrac{1}{2}(j^2 - s^2 - 3i^2) + 2is \qquad \text{and } d_l = \tfrac{1}{2}(i^2 + j^2) - 2(\tfrac{s}{2} + 1)^2 + 4(\tfrac{s}{2} + 1) - 2 \\
&= \tfrac{1}{2}(i^2 + j^2) - 2(\tfrac{s^2}{4} + s + 1) + 2s + 2 \\
&= \tfrac{1}{2}(i^2 + j^2 - s^2)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow d_t + d_l &= j^2 - s^2 - i^2 + 2is \\
&= j^2 - (i - j)^2 - i^2 + 2i(i - j) \\
&= j^2 - i^2 - j^2 + 2ij - i^2 + 2i^2 - 2ij \\
&= 0
\end{aligned}$$

Subcase$-$II : s is odd.

In this case, $n = i^2 + j^2$ is odd.

We prove that $d_t + d_l = 0$ and $t = i - \frac{s-1}{2}$, $l = \frac{s+1}{2}$.

Since n is odd, $d_1 = \frac{1}{2}(i^2+j^2-1)$, $d_2 = \frac{1}{2}(i^2+j^2-1)-4$, $\ldots$,$d_k = \frac{1}{2}(i^2+j^2-1)-2k(k-1)$.

$$\begin{aligned}
\Rightarrow d_t &= \tfrac{1}{2}(i^2 + j^2 - 1) - 2(i - \tfrac{s-1}{2})(i - \tfrac{s-1}{2} - 1) \\
&= \tfrac{1}{2}(i^2 + j^2 - 1) - 2[i^2 - \tfrac{(s-1)^2}{4} - i(s - 1) - i + \tfrac{s-1}{2}] \\
&= \tfrac{1}{2}(i^2 + j^2 - 1) - 2i^2 - \tfrac{1}{2}(s^2 - 2s + 1) + 2is - s + 1 \\
&= \tfrac{1}{2}(i^2 + j^2 - 1) - 2i^2 - \tfrac{s^2}{2} + s + 2is - s + \tfrac{1}{2} \\
&= \tfrac{1}{2}(i^2 + j^2) - 2i^2 - \tfrac{s^2}{2} + 2is
\end{aligned}$$

$$\begin{aligned}
\text{and } d_l &= \tfrac{1}{2}(i^2 + j^2 - 1) - 2(\tfrac{s+}{2})(\tfrac{s-1}{2}) \\
&= \tfrac{1}{2}(i^2 + j^2 - 1) - \tfrac{1}{2}(s^2 - 1) \\
&= \tfrac{1}{2}(i^2 + j^2 - s^2)
\end{aligned}$$

$$\Rightarrow d_t + d_l = i^2 + j^2 - s^2 - 2i^2 + 2is = j^2 - s^2 - i^2 + 2is = 0$$

Table-5 is computed according to above cases and it shows values of t and l to get $d_t + d_l = 0$. Thus, $\cup_{l=1}^{4} K_n(n = i^2 + j^2)$ is a cordial graph and so the index of cordiality for $K_n(n = i^2 + j^2)$ is at the most 4.

Illustration-2.2 **:** Index of cordiality for K_{80} is at the most 4.

The edge label difference sequence K_{80} is $d_1 = 40$, $d_2 = 38$, $d_3 = 32$, $d_4 = 22$, $d_5 = 8$, $d_6 = -10$, $d_7 = -32$, $d_8 = -58$ and $d_9 = -88$. According to *table*-5, $80 = 8^2 + 4^2$ and therefore, the value of $t = 8 - \lfloor \frac{4-1}{2} \rfloor = 7$, $l = \lceil \frac{4+1}{2} \rceil = 3$. Moreover $d_3 + d_7 = 0$. *Table*-6 establishes that union of four copies of K_{80} is cordial and so the index of cordiality for K_{80} is at the most 4.

Theorem-2.3 **:** If $D_j = 4$ in K_n, $D_{j-2} + 2D_j + D_{j+2} = 0$, and in this case, the index of cordiality for K_n is at the most 4.

Proof : According to discussion of cordiality of K_n,

$$\begin{aligned} D_j &= \tfrac{n-1}{2} - 2(j^2+1), && \text{when } n \text{ is odd.} \\ &= \tfrac{n}{2} - 2j^2, && \text{when } n \text{ is even} \end{aligned}$$

$$\begin{aligned} \Rightarrow D_{j-1} - D_{j-2} &= D_j - D_{j-1} + 4 = D_{j+1} - D_j + 8 \\ &= D_{j+2} - D_{j+1} + 12 \end{aligned}$$

$$\begin{aligned} \Rightarrow & D_{j-2} + D_j = 2D_{j-1} - 4, \\ & D_{j+2} + D_j = 2D_{j+1} - 4 \text{ and} \\ & D_{j+1} + D_{j-1} = 2D_j - 4. \end{aligned}$$

$$\begin{aligned} \Rightarrow D_{j+2} + 2D_j + D_{j-2} &= 2(D_{j+1} + D_{j-1}) - 8 \\ &= 2(2D_j - 4) - 8 \\ &= 2(2 \cdot 4 - 4) - 8 \\ &= 0 \end{aligned}$$

Since difference of vertex labels for D_j, D_{j-2} and D_{j+2}, D_j are precisely two, we get $v_f(0) = v_f(1)$ and $e_f(1) - e_f(0) = D_{j-2} + 2D_j + D_{j+2} = 0$ in $\cup_{j=1}^{4} K_n$. Thus $\cup_{j=1}^{4} K_n$ is cordial and the index of cordiality for K_n is at the most 4, when $D_j = 4$, for some i.

Illustration-2.4 **:** The index of cordiality for K_{44} is at the most 4, because according to *table*-1, $D_4 = d = 4$ and $D_2 + 2D_4 + D_6 = b + 2d + f = 20 + 8 - 28 = 0$.

Theorem-2.5 **:** If $D_j = 5$ in K_n, $D_{j-2} + D_{j-1} + D_{j+1} + D_{j+2} = 0$, and in this case, the index of cordiality for K_n is at the most 4.

Proof : By *Theorem*-2.3

$$D_{j+2} + D_{j-2} + 2D_j = 2(D_{j+1} + D_{j-1}) - 8$$
$$\Rightarrow D_{j+2} + D_{j-2} + D_{j+1} + D_{j-1} + 4 = 2(2D_j - 4) - 8$$
$$\Rightarrow D_{j+2} + D_{j+1} + D_{j-1} + D_{j-2} = 2(2 \cdot 5 - 4) - 8 - 4 = 0$$

Since difference of vertex labels for D_{j+2}, D_{j+1} and D_{j-1}, D_{j-2} are 1, we get $v_f(0) = v_f(1)$ and $e_f(1) - e_f(0) = D_{i+2} + D_{i+1} + D_{i-1} + D_{i-2} = 0$ in the union of four copies of K_n and so it is cordial. Thus, the index of cordiality for K_n is at the most 4, when $D_j = 5$, for some j.

Theorem-2.6 **:** If $D_j = 9$ in K_n, $D_{j-3} + 2D_j + D_{j+3} = 0$, and in this case, the index of cordiality for K_n is at the most 4.

Proof : According to proof of *Theorem*-2.3

$$\begin{aligned} D_{j-2} - D_{j-3} &= D_{j-1} - D_{j-2} + 4 = D_j - D_{j-1} + 8 \\ &= D_{j+1} - D_j + 12 = D_{j+2} - D_{j+1} + 16 \\ &= D_{j+3} - D_{j+2} + 20 \end{aligned}$$
$$\Rightarrow D_{j+3} - D_j = 3(D_{j-2} - D_{j-3}) - 48$$
$$D_j - D_{j-3} = 3(D_{j-2} - D_{j-3}) - 12$$
$$\Rightarrow D_{j+3} + D_{j-3} = -48 + 12 + 2D_j$$
$$\Rightarrow D_{j+3} + 2D_j + D_{j-3} = 4D_j - 36 = 0$$

Since difference of vertex labels for D_{j+3}, D_j and D_j, D_{j-3} are 3, we get $v_f(0) = v_f(1)$ and $e_f(1) - e_f(0) = D_{j-3} + 2D_j + D_{j+3} = 0$ in the union of four copies of K_n and so it is cordial. Thus, the index of cordiality for K_n is at the most 4, when $D_j = 9$, for some j.

n	a(1)	b(3)	c(5)	d(7)	e(9)	f(11)	g(13)	h(15)	i(17)		
5	2	-2	-10	-22						2a+2b	4
7	3	-1	-9	-21						a+2b	3
11	5	1	-7	-19						a+b+c	3
13	6	2	-6	-18						2a+2c	4
15	7	3	-5	-17	-33	-53				2b+c	3
17	8	4	-4	-16	-32	-52				2b+2c	4
19	9	5	-3	-15	-31	-51				2a+c+d	4
21	10	6	-2	-14	-30	-50				a+b+c+d	4
23	11	7	-1	-13	-29	-49				2b+d	3
27	13	9	1	-11	-27	-47				b+c+d	3
29	14	10	2	-10	-26	-46				2b+2d	4
31	15	11	3	-9	-25	-45				2a+c+d+e	5
33	16	12	4	-8	-24	-44				2d+a	3
35	17	13	5	-7	-23	-43				a+b+d+e	4
37	18	14	6	-6	-22	-42				2c+2d	4
39	19	15	7	-5	-21	-41				b+c+e	3
41	20	16	8	-4	-20	-40				2a+2e	4
43	21	17	9	-3	-19	-39				a+d+e	3
45	22	18	10	-2	-18	-38				2b+2e	4
47	23	19	11	-1	-17	-37	-61	-89	-121	b+d+e	3
51	25	21	13	1	-15	-35	-59	-87	-119	2a+e+f	4
53	26	22	14	2	-14	-34	-58	-86	-118	2c+2e	4
55	27	23	15	3	-13	-33	-57	-85	-117	a+2e	3
57	28	24	16	4	-12	-32	-56	-84	-116	c+2d+2e	5
59	29	25	17	5	-11	-31	-55	-83	-115	b+d+f	3
61	30	26	18	6	-10	-30	-54	-82	-114	2a+2f	4
63	31	27	19	7	-9	-29	-53	-81	-113	b+c+2e+f	5
65	32	28	20	8	-8	-28	-52	-80	-112	2d+2e	4
67	33	29	21	9	-7	-27	-51	-79	-111	a+e+f	3
69	34	30	22	10	-6	-26	-50	-78	-110	c+d+e+f	4
71	35	31	23	11	-5	-25	-49	-77	-109	b+e+f	3
73	36	32	24	12	-4	-24	-48	-76	-108	2c+2f	4
75	37	33	25	13	-3	-23	-47	-75	-107	a+b+f+g	4
77	38	34	26	14	-2	-22	-46	-74	-106	b+d+e+g	4
79	39	35	27	15	-1	-21	-45	-73	-105	c+d+f	3
83	41	37	29	17	1	-19	-43	-71	-103	d+2e+f	4
85	42	38	30	18	2	-18	-42	-70	-102	2d+2f	4
87	43	39	31	19	3	-17	-41	-69	-101	b+e+g	3
89	44	40	32	20	4	-16	-40	-68	-100	c+2e+g	4
91	45	41	33	21	5	-15	-39	-67	-99	c+e+g	3
93	46	42	34	22	6	-14	-38	-66	-98	a+c+f+h	4
95	47	43	35	23	7	-13	-37	-65	-97	b+2d+e+i	5
97	48	44	36	24	8	-12	-36	-64	-96	2c+2g	4
99	49	45	37	25	9	-11	-35	-63	-95	b+f+g	3
101	50	46	38	26	10	-10	-34	-62	-94	2e+2f	4
103	51	47	39	27	11	-9	-33	-61	-93	c+e+2f+g	5
105	52	48	40	28	12	-8	-32	-60	-92	d+e+f+g	4

Table−1

n	a(0)	b(2)	c(4)	d(6)	e(8)	f(10)	g(12)	h(14)	i(16)		
6	3	1	-5	-15						a+2b+c	4
8	4	2	-4	-14						2b+c	3
10	5	3	-3	-13						2c+2d	4
12	6	4	-2	-12						3b+d	4
14	7	5	-1	-11						a+b+c+d	4
18	9	7	1	-9	-23	-41				2a+2d	4
20	10	8	2	-8	-22	-40				2b+2d	4
22	11	9	3	-7	-21	-39				3b+d+e	5
24	12	10	4	-6	-20	-38				a+2c+e	4
26	13	11	5	-5	-19	-37				2c+2d	4
28	14	12	6	-4	-18	-36				2b+2c+2e	6
30	15	13	7	-3	-17	-35				b+c+d+e	4
32	16	14	8	-2	-16	-34				2a+2f	4
34	17	15	9	-1	-15	-33				2b+2e	4
38	19	17	11	1	-13	-31				c+2d+e	4
40	20	18	12	2	-12	-30				2c+2e	4
42	21	19	13	3	-11	-29				a+b+e+f	4
44	22	20	14	4	-10	-28				b+2d+f	4
46	23	21	15	5	-9	-27				b+c+e+f	4
48	24	22	16	6	-8	-26				b+2d+e+f	5
50	25	23	17	7	-7	-25				2d+2e	4
52	26	24	18	8	-6	-24	-46	-72	-102	2b+2f	4
54	27	25	19	9	-5	-23	-45	-71	-101	2b+e+g	4
56	28	26	20	10	-4	-22	-44	-70	-100	c+d+2e+f	5
58	29	27	21	11	-3	-21	-43	-69	-99	2c+2f	4
60	30	28	22	12	-2	-20	-42	-68	-98	2a+2d+2g	6
62	31	29	23	13	-1	-19	-41	-67	-97	a+b+f+g	4
64	32	30	24	14	0	-18	-40	-66	-96	2e	2
66	33	31	25	15	1	-17	-39	-65	-95	d+2e+f	4
68	34	32	26	16	2	-16	-38	-64	-94	2d+2f	4
70	35	33	27	17	3	-15	-37	-63	-93	c+3e+g	5
72	36	34	28	18	4	-14	-36	-62	-92	2a+2g	4
74	37	35	29	19	5	-13	-35	-61	-91	c+2e+g	4
76	38	36	30	20	6	-12	-34	-60	-90	4e+2f	6
78	39	37	31	21	7	-11	-33	-59	-89	a+c+f+h	4
80	40	38	32	22	8	-10	-32	-58	-88	2c+2g	4
82	41	39	33	23	9	-9	-31	-57	-87	2e+2f	4
84	42	40	34	24	10	-8	-30	-56	-86	c+d+e+f+2g	6
86	43	41	35	25	11	-7	-29	-55	-85	d+e+f+g	4
88	44	42	36	26	12	-6	-28	-54	-84	2e+4f	6
90	45	43	37	27	13	-5	-27	-53	-83	2d+2g	4
92	46	44	38	28	14	-4	-26	-52	-82	2c+2e+2h	6
94	47	45	39	29	15	-3	-25	-51	-81	c+d+e+f+i	5
96	48	46	40	30	16	-2	-24	-50	-80	4c+2i	6
98	49	47	41	31	17	-1	-23	-49	-79	2a+2h	4
100	50	48	42	32	18	0	-22	-48	-78	2f	2
102	51	49	43	33	19	1	-21	-47	-77	e+2f+g	4
104	52	50	44	34	20	2	-20	-46	-76	2e+2g	4

Table−2

Case number	Value of j	Status of n	Value of t	Value of l
1	i	even	i+1	1
2	i-1	odd	i	1
3	i-2	even	i	2
4	i-3	odd	i-1	2
General	i-s (i.e. s=i-j)	odd/even	i-⌊(s-1)/2⌋	i-⌈(s+1)/2⌉

Table−3

Here in each case $d_t + d_l$=0.

Order of copy in K_{80}	$v_f(0)$	$v_f(1)$	$e_f(1)$	$v_f(1)$	Diff.
1	42	38	1196	1564	32
2	38	42	1196	1564	32
3	46	34	1564	1196	-32
4	34	46	1564	1196	-32
Total	160	160	5520	5520	0

Table−4

4 Concluding Remarks :

In research the study of variety of graph labeling problems is the potential area. In this paper cordial labeling for union of some copies of the complete graph and the index of cordiality for the complete graph are discussed.

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