Riemannian Dynamics on Manifolds

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Abstract

This paper aimed at investigating the dynamical systems on manifolds, which is Riemannian dynamics 1-foliation $\mathcal{L}$ on 3-manifolds M [Carri`ere 17]. we explain that every point of a manifold $M$ is a recurrence point and the $w-limit$ sets are diffeomorphic to $M$. The structure of the attractor is also presented [15,10]. The map named after Hinri Poincare` on a transversal surface for a Riemannian 1-dimentional foliation is used as an isometry such that the nonhyperbolicity of $(M,\mathcal{L})$ is showed. our argument work for any dimension of a manifolds M.

Keywords: *Riemannian Dynamics, Limit sets, recurrence points and attractors Manifolds.*

I. INTRODUCTION

The purpose of this paper is to study dynamical systems on manifolds. We focus on dynamics of Riemannian 1-dimenstional foliations on 3-manifolds. The notions of hyperbolicity which is a key to understand the figure of the given 1-foliation, $w-limit\ sets$, recurrence and attractors of the foliations on the phase space are presented. If there is a Riemannian metric on the normal bundle in the sense of Carri`ere, then we call $\mathcal{L}$ is Riemmannian. We prove many dynamical properties of $(M,\mathcal{L})$ by using classification in [17] of all the oriented closed 3-manifolds equipped with a Riemannian 1-foliations. In ([1],[14]) our assumption is to consider that A nosov 1-foliation for a 1-foliation with respect to uniform hyperbolicity for a manifold M. See [1,16] for the concept of hyperbolic 1-foliations in generic dynamics. In particular, recurrence and $w-limit\ set$s are basic attributes in dynamical systems. In this paper, an important part is dealt with Conley's definition of attractors, see [5]. And we conclude the nonexistence of proper attractor for a Riemannian 1-foliation on closed 3-manifolds. In ([4], and [3]) the nontrivial attractor is mixing for a generic 3-dimenstion 1-foliations in compact 3-manifolds is proved, and the robust transitivity for 3-manifolds is described. Our study on the Riemannian 1-foliations on 3-manifolds is motivated by the classification theory of Brunella [12] and Ghys[7].

It is important to understand some motivation here, which comes from specific cases in [17]. Let T^nbe a n-dimensional torus. Recall that for any pair $(M,\mathcal{L})$ of a closed 3-manifolds and a Riemannian flow is one of the following:

1) M is a $T^3$and $\mathcal{L}$ is linear with an irrational slope on T^3.
2) M is also T^3. And $\mathcal{L}$ has two possibilities:
   2A) $M = T^2 \times S^1$. Let $\mathcal{L}^`$ be a linear 1-foliation on a $T^2$ with an irrational slope. Then $\mathcal{L}$ is the foliation such that for each $x \in S^1$,the induced flow $\mathcal{L}^t$ of $\mathcal{L}$ lies in $T^2 \times \{x\}$ and it coincides with $(\mathcal{L}^`)^t$.
   2B) fix any $c \in SL(2,\mathbb{Z})$ with eigenvalues $\lambda, \lambda^{-1} where\ \lambda > 1\ is\ defined\ to\ be\ an\ irrational\ number$. and the corresponding eigenvectors are $v, v^`$ respectively. Let $\mathcal{L}^`$ be the linear foliation on $\mathbb{R}^2$ whose time-1 map maps 0 to $v.\ and\ \mathcal{L}^`$ has an irrational slope. We consider a $\mathbb{Z}^2$-action comes from the standard affine translation group action on the first factor $\mathbb{R}^2$ and the $\mathbb{Z}^1$-action is defined by $n \cdot (m,t) = (c^n(m), t+n)$. The $(\mathbb{Z}^2 \times \mathbb{Z})$-quotient of $\mathbb{R}^3$ becomes a T^3. Let $T^3 = M$ and $\mathcal{L}$ be a 1-foliation on M as follows:
   Assume that $\mathcal{L}^{``}$ is a foliation such that for each $x \in \mathbb{R}^1$, its flow $(\mathcal{L}^{``})^t$ lies in $\mathbb{R}^2 \times \{x\}$ and it coincides with $(\mathcal{L}^`)^t$. Therefore $\mathcal{L}$ is set to the induced 1-foliation from $\mathcal{L}^{``}$ on the discrete group quotient M.
3) This case is divided as two parts:
   3A) M is a Lenz space $L(p,q)$, $(p,q \in \mathbb{Z}\backslash 0)$. At $p = 1\ and\ q = 0, L(1,0) \cong S^3$. M is defined as the quotient of the S^3:
   $S^3 = \{(z_1,z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ by a $\mathbb{Z}^1$-action
   $n \cdot (z_1,z_2) = (e^{2n\pi i/p} z_1, e^{2n\pi i/q} z_2)$. where
   $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.
   $S^3 \quad = |x_1 + iy_1|^2 + |x_2 + iy_2|^2 = 1$.

Let $\mu, \gamma \in \mathbb{R}\backslash\mathbb{Q}$ with $\gamma\backslash\mu \in \mathbb{R}\backslash\mathbb{Q}$ is the slope of $\mathcal{L}'_{(\gamma,\mu)}$.and $\mathcal{L}$ be the induced foliation on the discrete group quotient M from $\mathcal{L}'_{(\gamma,\mu)}$. let $\mathcal{L}'_{(\gamma,\mu)}$ be the foliation on S^3 whose corresponding flow is

$\mathcal{L}^t(z_1, z_2) = (e^{i\gamma t} z_1, e^{i\mu t} z_2), \quad t \in \mathbb{R}$.

3B) let $M = S^2 \times S^1 \cong L(0,1)$. Fixing the north and south poles of S^2 we assume that any flow on S^2given by rotation. Let $\mathcal{L}'$ be the corresponding foliation on S^2 and $\mathcal{L}$ be foliation such that for $x \in S^1$,its flow $\mathcal{L}^t$ lies in $S^2 \times \{x\}$ and it coincides with $(\mathcal{L}')^t$

4) M is a Seifert fibration, i.e., an S^1-fibration over a smooth 2-manifolds. And $\mathcal{L}$ is a foliation such that its flow $\mathcal{L}^t$lies in the fibre direction.
if M is Seifert manifold then $M(a,b) = L(b,a)\ and\ M((a_1,b_1),((a_2,b_2)) = L(a_1b_2 + a_2b_1, a_1a_2) = L(p,q)$.

Lemma (1.1). Let $(M^*, \mathcal{L}^*)$ be a pair of a smooth manifolds and a 1-foliation. Let $M = M^*/G$ be the group quotient such that G acts on M^*, and $\phi: M^* \to M$ is the quotient map. Suppose that there is a 1-foliation $\mathcal{L}$ on M and ϕ maps the flow $\mathcal{L}^{*t}$ to $\mathcal{L}^t$. then

i. The ϕ image of a recurrence point of $(M^*, \mathcal{L}^*)$ and any attractor of $(M, \mathcal{L})$ is given as the ϕ – image some attractor of $(M^*, \mathcal{L}^*)$.
ii. If G is a finite group, an $w - limit\ set$ of $(M, \mathcal{L})$ is the ϕ – image of some $w - limit\ set$ of $(M^*, \mathcal{L}^*)$.

Proof. The first case is clear so We will prove the case ii. Since ϕ is a covering map, $\phi^{-1}(A)$ is also an attractor. A is an attractor of $(M, \mathcal{L})$.
We have to take an $w - limit\ sets\ w(x)$ where $x \in M$. $y \in \phi^{-1}(x)$ and the ϕ – image of $set\ w(y)$coincides with $set\ w(x)$. Its clear that $\phi(w(y)) \subset w(x)$. There exist a sequence t_n with $Z = \lim_{n\to\infty} \mathcal{L}^{*t_n}(y)$ for any point $Z \in w(y)$.
$\lim_{n\to\infty} \mathcal{L}^{t_n}(x) = \phi(Z)$. We prove the $\phi(w(y)) \supset w(x)$. Let $Z \in w(x)$. Then there exist a sequence t_n with $\lim_{n\to\infty} \mathcal{L}^{t_n}(x) = Z$. By the unique lifting property of covering space of a path, $\mathcal{L}^t(x)$ lifts to $\mathcal{L}^{*t}(y)$ for any $y \in \phi^{-1}(x)$. There exist a subsequence t_{n_k} because a ϕ-fibre is a finite set. Since ϕ
Is a local isomorphism, so the ϕ – image of the limit coincides with Z.

Theorem (1.2). Every point of an oriented closed 3-manifolds is a recurrence point, if $\mathcal{L}$ is a Riemannian 1-foliation on M.

Proof. Our proof depends on [17]. Since any pair $(M, \mathcal{L})$ of a closed-manifold and a Riemannian flow is one of the cases[17] case (1), then all the positive orbits of the corresponding flow $\mathcal{L}^t$ of $\mathcal{L}$ are dense. Thus every point of $M$is a recurrence point.
Let $\mathcal{L}^*$be a linear 1-foliation on a 2-tours T^2 with an irrational slope. $M = T^2 \times S^1$. then $\mathcal{L}$ is the foliation such that for each $x \in S^1$, the induced flow $\mathcal{L}^*$ of $\mathcal{L}$ lies in $T^2 \times \{x\}$ and it coincides with $(\mathcal{L}^*)^t$. Case (2A) Therefore the $T^2 - fibration$ is trivial then this is easier.
We observe that M is a $T^2 - fibration$ over S^1.
We use the map $\phi: M \to \mathbb{R}^1/\mathbb{Z}$ induced by the projection $\mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^1$ where $M = \mathbb{R}^2 \times \mathbb{R}^1/\ \mathbb{Z}^2 \times \mathbb{Z}$ and the $\mathbb{Z} - action\ on\ \mathbb{R}^1$ is the standard translation action. $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ are fibres of the induced map. Each positive orbit closure of $\mathcal{L}$ becomes the $T^2 - fibre$ itself containing the orbit. Thus every point of M is a recurrence point.
In [17] case (3A). Assume that $p = q = 0$. $M = S^3$. For any $x \in T_k$ the flow $\mathcal{L}^t(x)$ lies in T_k. Where $0 \le k \le 1$.
$T_k = \{(z_1, z_2) \in S^3\ |z_1|^2 = k, \quad |z_2|^2 = 1 - k\}$.
$if\ T_k = S^1$ then $k = 0,1$
at $k = 0 \Longrightarrow \ T_0 = \{(z_1, z_2) \in S^3\ |z_1|^2 = 0, \quad |z_2|^2 = 1\}$,
$at\ k = 1 \Longrightarrow T_1 = \{(z_1, z_2) \in S^3\ |z_1|^2 = 1, \quad |z_2|^2 = 0\}$
Thus every point of T_k is a recurrence point.
By the assumption any positive orbit in T_k is dense in T_k because $\mathcal{L}^t$has an irrational slope ζ/η. Thus every point of M is a recurrence point.
In the case(3B) [17]. it is clear that every point is a recurrence point since the flow is induced from rotation of the S^2-factor.
In the case (4) [17]. $\mathcal{L}^t(x)$ lies in the $S^1 - fiber$ containing x for any $x \in M$, the positive orbit of each $p \in M$ is the $S^1 - fiber$ containing p. Thus p is a recurrence point.

Theorem (1.3). Let $(M, \mathcal{L})$ be an oriented closed 3-manifolds with a Riemannian 1-foliation. Then the $w-limit\ sets$ are diffeomorphic to either a circle S^1 or a T^2 or M itself. And there does not exist any proper nonempty attractor.

Proof. By a similar argument in [17] we shall proof our theorem. By the same reason in [17] case(1) the $w-limit\ sets$ of any point coincides with M itself.
Similarly no nonempty proper subset of M can be an attractor.
In [17] case (2A) is easier because the $T^2-fibration$ is trivial. Thus a similar proof works as in the case(2B) [17].
From [17] case(2B) we also deduce that the $w-limit\ sets$ of any point is the $T^2-fibre$ containing the point. In this case we prove that there are no non empty proper attractors. Let A be a nonempty proper attractor if any. Let U be an attractor block of A. And V be an open subset in S^1 such that $\varphi|_{\varphi^{-1}(v)}$ gives a trivial $T^2-fibration\ \varphi^{-1}(v)$ intersects A but is not contained in A.
Suppose that $A^{`}=(\varphi^{-1}(v)\cap A)$ and $U^{`}=((\varphi^{-1}(v)\cap U)$. Let $p\in A^{`}$. since any positive orbit lies in a $T^2-fibre$, $A^{`}$ contains the $w-limit\ w(p)$. For some proper relatively closed subset B in V, $A^{`}$ is the form $T^2\times B$.
$A^{`}=\bigcap_{t\geq 0}\overline{\mathcal{L}^t(U)}=T^2(\bigcap_{t\geq 0}\overline{\mathcal{L}^t(V^{`})})=T^2\times V^{`}$ then there exists a proper open subset $V^{`}\ in\ V$ with $B\subset V^{`}$. For any $t\geq 0, \mathcal{L}^t(V^{`})=V^{`}$ since the flow is parallel to the $T^2-fibres$. Since $T^2\times V^{`}$ properly contains $A^{`}$ this is a contradiction. Since the $T^2-fibration$ is trivial therefore case (2A) in[17] is easier. In [15] case (3A) assume that $p=q=0$ then $M=S^3$.
$T_k=\{(z_1,z_2)\in S^3\mid |z_1|^2=k, |z_2|^2=1-k\}$ where $0\leq k\leq 1$. The flow $\mathcal{L}^t(x)$ lies in T_k for any $x\in T_k$. By the similar argument in the theorem(1.1) the $w-limit$ set is T_k itself . if $k=0,1$ then $T_k=S^1=|x+iy|^2=x^2+y^2=1$ and it coincides with the positive orbit. If $0<k<1$ then $T_k=T^2=S^1\times S^1$ and any positive orbit in T_k is dense in T_k because $\mathcal{L}^t$ has an irrational slope $\lambda\backslash\mu$. Therefore for every point of T_k it $w-limit$ set is T_k itself. We show that there is no nonempty proper attractor. Suppose not, so there exists a nonempty proper attractor A. Let U be an attractor block of A. $\varphi: S^3\to[0,1]$, $(z_1,z_2)\longrightarrow|z_1|^2$ restricts to $T^2-fibration$ over the smaller base $(0,1)$. $T_k=\varphi^{-1}(k)$. Let $A\cap\ T^2-fibration=A^{`}$ and $U\cap T^2-fibration=U^{`}$ is nonempty then $U^{`}$ is also an attractor block of $A^{`}$ with respect to the restricted foliation. $\dim U^{`}=3$ but $M\backslash\varphi^{-1}(0,1)=T_0\sqcup T_1$ has dimension1. $A^{`}$is empty. By a similar argument in (2B) [17], $\bigcap_{t\geq 0}\overline{\mathcal{L}^t(U^{`})}$ properly contains $A^{`}$ because φ restricts to the trivial fibration in our case.
By assumption $(p,q)\neq(0,0)$. The result holds after passing to the quotient due to lemma (1.1) then M is a finite group quotient of S^3. In[17] case(3B) if $p\in M=S^2\times S^1$ projects to the north or south pole of S^2, $w(p)\ is\ S^1$. Other wise it is T^2. By a similar argument of the case $p=q=0$ there is no nonempty proper attractor. In [17] case (4) for any $x\in M$ implies $\mathcal{L}^t(x)$ lies in the $S^1-fibre$ containing x, the positive orbit M is the S^1-fibre for every $point\ p\in M$.therefore $w(p)$ is the $S^1-fibre$. So there is no proper attractor. This complete the proof.
In the case $H^2(M,\mathcal{O})=0$, Brunella classifies all the pairs $(M^*,\mathcal{L}^*)$ where $M^*and\ \mathcal{L}^*$ are obtained by (complexification of leaves) of Haefliger-Sundararman [2, proposition 2.1].

II. BASIC PROPERTIES, NOTATIONS AND DEFENTIONS.

Let $\mathcal{L}:\mathbb{R}\times M\to M$ be a 1-foliation on Hausdorff topological space M. Since $M\ is$ Hausdorff then for each $a,b\in M$ there exist two open sets $H\ and\ G$ such that $a\in G\ and\ b\in H,\ \ G\cap H=\emptyset$.
$\mathcal{L}^t=\mathcal{L}(t,\cdot):M\to M$ denotes short. We dine
$w(q)=\{x\in M: x=\lim_{n\to\infty}\mathcal{L}^{t_n}(q)$ for some sequence $t_n\to\infty\ as\ n\to\infty\}$ is $the\ w-limit\ set\ of\ q$.
and $\beta(q)=\{x\in M: x=\lim_{n\to\infty}\mathcal{L}^{-t_n}(q)$ for some sequence $t_n\to\infty\ as\ n\to\infty\}$ is $\beta-limit\ set\ of\ q$.

Definition (2.1). A point $x\in M$ is positively recurrent or $w-recurrent$ with respect to $\mathcal{L}^t$ if $x\in w(x)$ and is $\beta-recurrent$ or negatively recurrent with respect to $\mathcal{L}^t$ if $x\in\beta(x)$. If x is simultaneously positively and negatively recurrent then a point $x\in M$ is (poincarè) recurrent with respect to $\mathcal{L}^t$.

Definition (2.2). Let $\mathcal{L}:\mathbb{R}\times M\to M$ be a 1- foliation on a finite dimensional smooth manifold M. A compact $\mathcal{L}-invirant$ set, $\Lambda\subset M$ is called a hyperbolic set for the 1- foliation $\mathcal{L}$ if there exisit $c>0$ and $0<\lambda<1$ therefore for each $x\in\Lambda$, there exists a decomposition $T_xM=E_x^{ss}\oplus E_x^{uu}\oplus E_x^c$
$\partial_t\mathcal{L}(t,x)|_{t=0}\in E_x^c-\{0\},\ dim(E^c(x))=1,$

$D_t\,\mathcal{L}(t,x)(E_x^i) = E_x^i$ with $i = ss, uu, and$ $\|D_t\,\mathcal{L}(t,x)\,|\,E_x^{uu}\| \le c\lambda^{\,t}\ for\ t \le 0$. $\|D_t\,\mathcal{L}(t,x)\,|\,E_x^{ss}\| \le c\lambda^{\,t}\ for\ t \ge 0$. Such that $\|\cdot\|$ Is a nom induced by the Riemannian metric. The $c^{`}$ 1-foliation $\mathcal{L}$ is called an Anosov 1-foliation if the whole manifolds M is a hyperbolic set for $\mathcal{L}$. The bundle E^{ss} is called strong stable and the bundle E^{uu} is called strong unstable bundle of $\mathcal{L}$ with an Anosov 1-foliation $\mathcal{L}$ on a compact connected manifold M. So a derivative $D_x\,\mathcal{L}$ of a 1-foliation $\mathcal{L}$ is eventually contracting on E_x^{ss} and expanding on E_x^{uu}.

Definition (2.3). Let (X,d) be a metric space and φ^t be a continuous 1-foliation on X. A nonempty open subset U of X is an attractor block for $\overline{\varphi^t(U)} \subseteq U$ for every $t > 0$. If there exist an attractor block U satisfying $A = \bigcap_{t\ge 0}\overline{\varphi^t(U)}$ then a proper $A \in X$ is called an attractor for φ^t. Let $\mathcal{L}$ be a nondegenerate 1-foliation on $M, i.e.,$ a nowhere vanishing smooth section of the tangent bundle TM. M is smooth manifolds. The corresponding 1-foliation $\mathcal{L}^t$ has nowhere vanishing derivative with respect to t. From our assumption the argument works for any dimension of a manifolds M. Let D_x be any embedded unit disk in M centred at x. $x \in M$ is a fixed point. We suppose that any tangents of D_x are transverse to $\mathcal{L}$. $\mathcal{L}^{t_0}(x) \in D_x$ with positive t_0 but $\mathcal{L}^t(x) \notin D_x$ for any $t \in (0,t_0)$. There exist an open neighbourhood $U\ of\ x$ in D_x such that the diffeomorphism of the Poincare$^{`}$ map ρD_x assingns $y \in U$ to the first touching point $\mathcal{L}^t(y)$ in $\mathcal{L}^t(y)$, $t > 0$ in D from U onto the image. $\rho D_x : U \to D_x$. let D_0 be the unit disk in $\mathbb{R}^2$ and $C_x : D_0 \to D_x$ me the diffeomorphism for the above embedded disk D_x for each $x \in M$. Let . $\epsilon_x > 0$ such that . ι_x extends to an open embedding $\tilde{\iota}_x = D_0 \times (-\epsilon_x, \epsilon_x) \to M$. $\tilde{\iota}_x(y,t) = \mathcal{L}^t(\iota_x(y)),\ t \in (-\epsilon_x, \epsilon_x)$ for all $y \in D_0$. the $(-\epsilon_x, \epsilon_x)$- direction is parallel to the 1-foliation of $\mathcal{L}$. The set $\{U_x\}_{x \in M}$ is an open cover of M. U_x denotes the $\tilde{\iota}_x$-image open subset of M. Let δ_{xy} be the composite $p_x \circ \iota_y$. $p_x : U_x \to D_0$ is the composite of $\tilde{\iota}_x \mid_{p_x \circ \iota_y} U_x^{-1}$ with the projection to D_0 . the composition is not well- defined over D_0 but only over the subset $\iota^{-1}{}_y(U_x)$. By shrinking domains the compostion $\delta_{zx} \circ \delta_{xy} = \delta_{zx}(\delta_{xy}) = p_z(\iota_x\left(p_x(p_y)\right))$ coincides with δ_{zy} over an open subset in D_0 . we have the group law in the set $\Gamma = \{\delta_{xy}\}_{x,y\in M} = \{p_x(p_y)\}$.
With the identity and inverses. This called a holonomy pseudo-group.

Lemma (2.4). By shrinking domains $p_z = p_z(\iota_x(p_x))$.
Proof. p_z maps $\ddot{x}$ to another intersection point in D_z with the leaf containing $\ddot{x}$ Both leaves are same as they contain $\ddot{x}$ **.** Therefore, we obtain the equivalence of the maps.

Definition (2.5). $\mathcal{L}$ is Riemannian if there exist embeddings $\iota_x : (D_0, 0) \to (M, x),\ x \in M$ and Riemannian metric on D_0 invariant under all the locally defined diffeomorphism in the associated pseudo- group Γ. Since $\Gamma = \{\delta_{xy}\}_{x,y\in M}$ this called a (holonomy) pseudo-group.

Lemma (2.6). If $\mathcal{L}$ is Riemannian then there is the induced smoothly varying fibre-wise metric g on $t \times D_0$ for each $t \in (\epsilon_x, \epsilon_x)$.

Definition (2.7). We identify $\mathbb{R}^2$ with $\mathbb{C}$ and D_0 is an analytic open subset. $\mathcal{L}$ is transversely holomorphic if it is nondegenerate and there exist embeddings $\iota_x : (D_0, 0) \to (M, x),\ x \in M$ satisfying
δ_{xy} of the pseudo-group Γ are all holomorphic maps.

III. NONHYPERBOLICITY AND TRANSVERSELY HOLOMORPHIC 1-FOLIATIONS.

Definition (3.1). Let (M,f) be a pair of a finite dimensional smooth manifold and a diffeomorphism on M. M has a hyperbolic structure with respect to of f if there exist a Riemannian metric on M and continuous aplitting of TM into the direct sum of Tf- invariant subbundles $E^s and\ E^u$ such that for constants A and λ, $0 < \lambda < 1$ and for $n = 0$,
$\|T\,f^0(v)\| \le A\,\lambda^0\,\|v\|$, $\|T\,f^0(w)\| \le A\,\lambda^0\,\|w\|$ for all $v \in E^s, w \in E^u$ and $\|\cdot\|$ is a norm induced by the Riemannian metric. where $n > 0$ the inequalities will be
$\|T\,f^n(v)\| \le A\,\lambda^n\,\|v\|$, $\|T\,f^n(w)\| \le A\,\lambda^n\,\|w\|$ (3.1)

A hyperbolic subset of M with respect to f is a closed invariant subset of M with hyperbolic structure and restriction of f.

Lemma (3.2). Let $(M,\mathcal{L})$ be a Riemannian 1- foliation on an oriented closed 3-manifold M. the poincare' map $\rho\ D_x$ on the embedded disk $D_x = \iota_x(D_0)$ for any $x \in M$ is an isometry. Hence it is nonhyperbolic.

Proof. $\rho\ D_x$ is not vacuous for some x. Because the generality of the compactness of M implies the existence of recurrent point (see [11, p.101]). Since every point of M is a recurrent point thus for any $x \in M$, $\rho\ D_x$ is defined. By lemma (2.6) the induced metric D_x does not rely on x. So $\rho\ D_x$ is an isometry. From isometry the nonhyperbolicity is clear. The proof is completed.

Before we apply the next theorem let us take a sheaf $\mathcal{O}$ of germs of function on M which are constant along the leaves holomorphic in the transverse direction.

Theorem (3.3). ([Ghy's theorem[7]) Let $\mathcal{L}$ be a transversely holomorphic foliation on M. If $H^2(M,\mathcal{O}) \neq 0$, $\mathcal{L}$ is Riemannian. M is closed 3-manifold.

Theorem (3.4). Let $(M,\mathcal{L})$ be as in theorem (3.2). then $(M,\mathcal{L})$ is nonhyperbolic with respect to any Riemannian metric M.

Proof. Since a tubular neighbourhood of the path defining ρD_x has the induced Riemannian metric on M as it is diffeomorphic to $\mathcal{N}$ along the path.

We can set up a metric on M from this metric such that its restriction to a smaller tubular neighborhood of the path still has the same induced metric. This is obtained by the partition of unity. By lemma (3.2), ρD_x is nonhyperbolic and thus so is $(M,\mathcal{L})$ with respect to the metric. In general, (3.1) is independent of the choice of metric. $(M,\mathcal{L})$ is also nonhyperbolic with any Riemannian metric.

Corollary (3.5). Let $\mathcal{L}$ be a transversely holomorphic foliation on a closed 3-manifold M. If $H^2(M,\mathcal{O}) \neq 0$, the poincare' map ρD_x for each recurrent point x is nonhyperbolic with respect to the induced Riemannian metric. So ρD_x is nonhyperbolic with any Riemannian metric.

Now we take transversely holomorphic 1-foliations instead of Riemannian foliations and complexification of sheaf $\mathcal{O}$ on M. We need to apply Brunella and Ghy's works [12,2]. In [8, propostion 4.1] we will obtain the next corollary.

Corollary (3.6). If $\mathcal{L}$ is a transversely holomorphic 1-foliation, there is $\tilde{\iota}_{x_i}, x_i \in M^n$. $\delta_{x_i y_j}$ is holomorphic and $\tilde{h}_{x_i y_j}$ is harmonic so that: $\tilde{\iota}^{-1}{}_{x_i}\left(\tilde{\iota}_{y_j}\right): D_{x_i y_j} \times \left(-\epsilon_{y_j}, \epsilon_{y_j}\right) \to D_{x_i y_j} \times \left(-\epsilon_{x_i}, \epsilon_{x_i}\right)$,

$(z,t) \ \to (\delta_{x_i y_j}(z), t + \tilde{h}_{x_i y_j}(z))$.

Definition (3.7). Let $\mathcal{O}$ be the sheaf on M,we will take all of the sheave along $\mathcal{O}_M^\infty$ such that each open subset U_i of M^n, $\mathcal{O}_M^\infty(U_i)$ is the set of functions on U_i and $f_i \circ \tilde{\iota}_{x_i}|\tilde{\iota}_{x_i\,(u_{x_i} \cap u_i)}$ is holomorphic on D_0 and constant a long $(-\epsilon_{x_i}, \epsilon_{x_i})$for all $x_i \in M_\infty^n$.

Since the compact manifolds can be covered by finitely many coordinate charts, we take many x_i such that u_{x_i} forms an open cover of M. Let $\epsilon = min_i(\epsilon_{x_i})$. Let $\tilde{m}_{ij}$ be harmonic conjuagate of $\tilde{h}_{ij}$ such that $\tilde{h}_{ij} + i^* \tilde{m}_{ij} : (-\epsilon,\epsilon) \times S^1 \to \mathbb{C}$ is holomorphic such that $(-\epsilon,\epsilon) \times S^1$is a ring in $\mathbb{C}$.

$\hat{\delta}_{ij}$: $D_{ij} \times : (-\epsilon,\epsilon) \times S^1 \to \quad D_{ij} \times (-\epsilon,\epsilon) \times S^1$,

$(z,t,s) \quad \to \ (\delta_{ij}(z), t + \tilde{h}_{ij}(z), s + \tilde{m}_{ij}(z))$

(see [12, p.276]).

Generally $\hat{\delta}_{im} = \hat{\delta}_{ij} \circ \hat{\delta}_{jm} \quad (*)$ where $\hat{\delta}_{ij}$ do not satisfy 1-cocycle condition$(*)$. If $H^2(M,\mathcal{O}) \neq 0$, $\mathcal{L}$ is Riemannian by [7, theorem 1.1]. from [12, proposition3] if $H^2(M,\mathcal{O}) = 0$, they form a 1-cocycle.

Corollary (3.8). Let $(M,\mathcal{L})$ be a Riemannian 1-foliation transversely holomorhic on an oriented closed 3-manifold. If $H^2(M,\mathcal{O}) \neq 0$ then $(M,\mathcal{L})$ is nonhyperbolic with respect to any Riemannian metric on M.

Corollary (3.9). By theorem(3.3) if $H^2(M,\mathcal{O}) \neq 0$, then the w-limit sets are diffeomorphic to S^1 or T^2or M. And does not exist any proper nonempty attractor.

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