# On The Application of Construction Method in Advanced Algebra

Yang He<sup>#1</sup>, Yanhui Zhai<sup>\*2</sup>

<sup>#</sup>School of Mathematical Science, TianGong University, Tianjin 300387, China

**Abstract** — In many ways to solve mathematical problems, "construction law" is a very clever innovative method, construction method is widely used in solving problems in advanced algebra, it can effectively solve some difficult problems in mathematics, in this paper, through advanced algebra problems of concrete instance, the application of construction method in solving high generationproblems is summarized. It is helpful to help us master the structural method, at the same time it improves the quality of mathematics and cultivate the consciousness of innovation.

Keywords — Construction method , Polynomial, Matrix, Determinant

## I. INTRODUCTION

In some math problems, sometimes a mathematical model is constructed to solve mathematical problems (Such as geometric figures, functions, equations, etc), to find some kind of intrinsic connection in the mathematical problem, through a certain connection in the problem, the mathematical problem can be made simple and clear, so as to play the role of transformation and bridge, and then find the way of thinking and method to solve the mathematical problem, this problem solving method is called construction method.<sup>[11]</sup> Construction method has been a common method of solving problems in high school mathematical problem solving break the convention, find a new way and get the solution skillfully. This paper mainly combines the concrete examples of advanced algebra problems and the specific application of the categorization and summary construction method in solving the problem of high generation .Let everybody master construction method proficiently, improve mathematical quality and cultivate innovation consciousness.

### II. THE CONSTRUCTIONOFPOLYNOMIAL PROBLEMS

*Case A*Prove that any rational coefficient polynomial with degree greater than zero can be represented as the sum of irreducible polynomials in the field of rational Numbers<sup>[5]</sup>.

Analysis: This problem is a qualitative decomposition of arbitrary polynomials, so it is necessary to qualitatively construct the expression in the conclusion. The key of the problem is how to construct these two irreducible polynomials with rational coefficients. However, our common polynomial with irreducible rational coefficients is a polynomial of first degree and a polynomial of rational coefficients satisfying the conditions of Eisenstein discriminant method.

**Poof:** A.if 
$$f(x) \in z[x]$$
, set  $f(x) = \sum_{i=0}^{n} a_i x^i$ , (where  $a_n \neq 0, n \ge 1$ ).

1) If  $a_0 = 0$ , takes prime number p, constructs polynomial: t(x) = pf(x) + x' + p (s > n).

According to eisenstein's criterion, t(x), h(x) = x' + p is irreducible in the domain of rational

Numbers, So  $f(x) = \frac{1}{p}t(x) - \frac{1}{p}h(x)$  is also irreducible in the domain of rational Numbers.

2) If  $a_0 \neq 0$ , taking prime number P cannot be divisible into  $a_0$ , (p > 2), constructs polynomial : t(x) = pf(x) + x' + p, (p-2),  $a_0(s > n)$ , the constant term of t(x) is  $pa_0 + p(p-2)a_0 = pa_0(p-1)$ . According to eisenstein's test, t(x), h(x) = x' + p, (p-2),  $a_0$  is irreducible in the rational number domain, so  $f(x) = \frac{1}{p}t(x) - \frac{1}{p}h(x)$  is irreducible in the rational number domain.

**B.** If there is  $f(x) \in Q[x]$ , then there is  $m \in z$ , such that  $mf(x) \in Z[x]$ . From (I), we can know: there is an irreducible polynomial  $u(x), v(x) \in Q[x]$  in the field of rational Numbers, so that mf(x) = u(x) + v(x), therefore  $f(x) = \frac{1}{m}u(x) + \frac{1}{m}v(x)$ , where  $\frac{1}{m}u(x)$ ,  $\frac{1}{m}v(x)$  is an irreducible polynomial in the field of rational Numbers.

In solving polynomial problems, the construction method mainly constructs the polynomial that meets the requirements by using the known conditions and the properties of polynomials, and finally obtains the solution.

### **III. THE CONSTRUCTION OF DETERMINANT PROBLEMS**

Case B Compute the nth order determinant 
$$D = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \cdots & \cdots & \cdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ x_1^n & x_2^n & \cdots & x_n^n \end{vmatrix}$$

Analysis: At first glance, this determinant looks like A vandermonde determinant, but in fact it is not, because the last row is not  $x_1^{n-1}, x_2^{n-1}, x_3^{n-1}, \dots, x_n^{n-1}$ . However, we can solve the problem by constructing a n+1 van der monde determinant, which can be solved as follows:

Solution: Structural determinant : 
$$D_{1} = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ x_{1} & x_{2} & \cdots & x_{n} & x_{n+1} \\ x_{1}^{2} & x_{2}^{2} & \cdots & x_{n}^{2} & x_{n+1}^{2} \\ \cdots & \cdots & \cdots & \cdots \\ x_{1}^{n-1} & x_{2}^{n-1} & \cdots & x_{n}^{n-1} & x_{n+1}^{n-1} \\ x_{1}^{n} & x_{2}^{n} & \cdots & x_{n}^{n} & x_{n+1}^{n} \end{vmatrix}$$

so the subexpression of the cofactor of  $x_{n+1}^{n-1}$  in the expansion of determinant  $D_1$  is  $(-1)^{n+(n+1)}D$ , and

$$D_{1} = \prod_{1 \le j < i \le n+1} (x_{i} - y_{j}) = (x_{n+1} - y_{1})(x_{n+1} - y_{2})\cdots(x_{n+1} - y_{n})\prod_{1 \le j < i \le n} (x_{i} - y_{j})$$
$$= \left[x_{n+1}^{n} - (x_{1} + x_{2} + \cdots + x_{n})x_{n+1}^{n-1} + \cdots + (-1)^{n}x_{1}x_{2}\cdots x_{n}\right]\prod_{1 \le j < i \le n} (x_{i} - x_{j})$$

so 
$$D = (x_1 + x_2 + \dots + x_n) \prod_{1 \le j < i \le n} (x_i - x_j)$$
.

The construction method has some difficulties in solving determinant problems, mainly by using the known conditions, using the basic properties of determinant and some special determinants such as van der monde determinant to construct special row and column formulas or simplify the original determinant, and finally solving the problem effectively.

#### IV. THE CONSTRUCTION OF QUADRATIC PROBLEMS

*Case CLet A* is the n-order symmetric matrix in the real number domain, and verify: there is real d, such that for any n dimensional column vector X in the real domain, there is  $|X'AX| \le dXX'$ , where X' is the transpose matrix of  $X^{[5]}$ .

**Poof:** Since A is a n-order symmetric matrix of order, construct n quadratic form  $f(x_1, \dots, x_n) = X'AX$ .

Then, there is an orthogonal linear change of X = QY, which can transform the above quadratic form into the standard form  $f(x_1, x_2, \dots, x_n, ) = X'AX = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$ .

If  $d = \max \{ |\lambda_1|, |\lambda_2|, \dots, |\lambda_n| \},$ so  $-dX'AX = -dY'Q'QY = -dY'Y = -d(y_1^2 + y_2^2 + \dots + y_n^2) \le \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2 \le d(y_1^2 + y_2^2 + \dots + y_n^2) \le dY'Y = dX'X, so|X'AX| \le dXX'.$ 

In solving the quadratic problem, the construction method mainly constructs the quadratic or symmetric matrix conforming to the conditions by virtue of the known conditions and the properties of the quadratic and symmetric matrix, and finally solves the problem.

#### V. THE CONSTRUCTION OF MATRIX PROBLEMS

**Case DL**et A is a *n* order square in field P, and E is a nth order identity matrix, proving that  $A^2 = E$  is only if and of r(A+E) + r(A-E) = n.

*Poof:* Construct a partitioned matrix  $\begin{pmatrix} A+E & 0\\ 0 & A-E \end{pmatrix}$ , and you transform it, you get

$$\begin{pmatrix} A+E & 0\\ 0 & A-E \end{pmatrix} \xrightarrow{r_2+r_1} \begin{pmatrix} A+E & 0\\ A+E & A-E \end{pmatrix} \xrightarrow{c_1-c_2} \begin{pmatrix} A+E & 0\\ 2E & A-E \end{pmatrix} \xrightarrow{r_1-\frac{1}{2}(A+E)r_2} \\ \begin{pmatrix} 0 & -\frac{1}{2}(A^2-E)\\ 2E & A-E \end{pmatrix} \xrightarrow{-2r_1} \begin{pmatrix} 0 & A^2-E\\ 2E & A-E \end{pmatrix} \xrightarrow{c_2-\frac{1}{2}Ac_1} \begin{pmatrix} 0 & A^2-E\\ 2E & -E \end{pmatrix} \xrightarrow{c_2+\frac{1}{2}c_1} \begin{pmatrix} 0 & A^2-E\\ E & 0 \end{pmatrix}$$

so 
$$r(A+E) + r(A-E) = n + r (A^2 - E)$$
,

so 
$$r(A+E) + r(A-E) = n$$
 if and only if  $r(A^2 - E) = 0$ , that is  $A^2 = E$ .

In solving the matrix problem, the construction method mainly constructs the appropriate partitioned matrix by using the known conditions and the properties of the matrix, which can bring great convenience for solving the problem.

#### VI. THE CONSTRUCTION OF LINEAR SPACE PROBLEMS

*Case ELet V* be an n-dimensional linear space on  $R(n \ge 1)$ , and prove the existence of an infinite subset W of V, making any n vectors in theta linearly independent.

**Poof:** Let  $\boldsymbol{\alpha}_1, \ \boldsymbol{\alpha}_2, \ \boldsymbol{\alpha}_3, \dots, \boldsymbol{\alpha}_n$  be a basis for V, then  $V = L(\boldsymbol{\alpha}_1, \ \boldsymbol{\alpha}_2, \ \boldsymbol{\alpha}_3, \dots, \ \boldsymbol{\alpha}_n)$ , so  $\forall \boldsymbol{\beta} \in V, \boldsymbol{\beta} = k_1 \boldsymbol{\alpha}_1 + k_2 \boldsymbol{\alpha}_2 + k_3 \boldsymbol{\alpha}_3 + \dots + k_n \boldsymbol{\alpha}_n$ , in particular, let's construct the following vector  $\boldsymbol{\beta}_{i_k} = i_k \boldsymbol{\alpha}_1 + i_k^2 \boldsymbol{\alpha}_2 + i_k^3 \boldsymbol{\alpha}_3 + \dots + i_k^n \boldsymbol{\alpha}_n$ ,  $i, k \in N^+$ ,

$$\operatorname{so}\begin{pmatrix}\boldsymbol{\beta}_{i_{1}}\\ \boldsymbol{\beta}_{i_{2}}\\ \vdots\\ \boldsymbol{\beta}_{i_{n}}\end{pmatrix} = \begin{pmatrix} i_{1} & i_{1}^{2} & \cdots & i_{1}\\ i_{2} & i_{2}^{2} & \cdots & i_{2}^{n}\\ \cdots & \cdots & \cdots\\ i_{n} & i_{n}^{2} & \cdots & i_{n}^{n} \end{pmatrix} \begin{pmatrix}\boldsymbol{\alpha}_{1}\\ \boldsymbol{\alpha}_{2}\\ \vdots\\ \boldsymbol{\alpha}_{n} \end{pmatrix}, \operatorname{let} A = \begin{pmatrix}\boldsymbol{\beta}_{i_{1}}\\ \boldsymbol{\beta}_{i_{2}}\\ \vdots\\ \boldsymbol{\beta}_{i_{n}} \end{pmatrix}, B = \begin{pmatrix} i_{1} & i_{1}^{2} & \cdots & i_{1}\\ i_{2} & i_{2}^{2} & \cdots & i_{2}^{n}\\ \cdots & \cdots & \cdots\\ i_{n} & i_{n}^{2} & \cdots & i_{n}^{n} \end{pmatrix}, C = \begin{pmatrix}\boldsymbol{\alpha}_{1}\\ \boldsymbol{\alpha}_{2}\\ \vdots\\ \boldsymbol{\alpha}_{n} \end{pmatrix}$$

that A = BC. Because  $i_k \neq i_j \neq 0$ , the van der monde determinant tells us

$$|B| = \begin{vmatrix} i_1 & i_1^2 & \cdots & i_1^n \\ i_2 & i_2^2 & \cdots & i_2^n \\ \cdots & \cdots & \cdots & \cdots \\ i_n & i_n^2 & \cdots & i_n^n \end{vmatrix} = \prod_{k=1}^n i_k \begin{vmatrix} 1 & i_1^2 & \cdots & i_1^{n-1} \\ 1 & i_2^2 & \cdots & i_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & i_n^2 & \cdots & i_n^{n-1} \end{vmatrix} \neq 0, \text{ so } r(B) = n.$$

Due to  $\boldsymbol{\alpha}_1$ ,  $\boldsymbol{\alpha}_2$ ,  $\boldsymbol{\alpha}_3$ ,  $\cdots \boldsymbol{\alpha}_n$  is a basis for V, so r(C) = n. Let  $W = \{ \boldsymbol{\beta}_{i_k} | \boldsymbol{\beta}_{i_k} = i_k \boldsymbol{\alpha}_1 + i_k^2 \boldsymbol{\alpha}_2 + i_k^3 \boldsymbol{\alpha}_3 + \cdots + i_k^n \boldsymbol{\alpha}_n \},\$ 

 $i, k \in N^+$ . Obviously, W is an infinite subset of V, and any n vectors in W are linearly independent.

In solving the problem of linear space, the construction method mainly constructs the vector that meets the condition by using the known conditions and the properties of the linear space basis, so as to achieve the purpose of solving the problem.

#### VII. THE CONSTRUCTION OF LINEAR TRANSFORMATION PROBLEMS

*Case FLet* V be an n-dimensional linear space, and prove that any subspace W of V must be the kernel of some linear transformation.

**Poof:** 1) When  $W = \{0\}$ , W is the kernel of the identity transformation;

2) When W = V, W must be the kernel of the zero transformation;

3) When  $\{0\} \subset W \subset V$ , That is, if W is the true subspace of V, set dim(W) = r,

Take a group of basis  $\alpha_1$ ,  $\alpha_2$ ,  $\alpha_3$ ,  $\cdots$ ,  $\alpha_r$  of W and extend it to a group of basis  $\alpha_1$ ,  $\alpha_2$ ,  $\alpha_3$ ,  $\cdots$ ,  $\alpha_r$ ,  $\alpha_{r+1}$ ,  $\cdots$ ,  $\alpha_n$  of V.

Construct linear transformation:  $\sigma(\alpha_i) = \begin{cases} 0 & (1 \le i \le r) \\ \alpha_i & (r \le i \le n) \end{cases}$ , Obviously,  $\sigma$  is a linear transformation, and

W is the kernel of the linear transformation  $\sigma$  .

In solving the linear transformation problem, the construction method mainly constructs the linear transformation and matrix which accord with the conditions by using the known conditions and the isomorphism and properties of linear transformation and matrix, and then solves the problem.

#### VIII. CONCLUSION

Construction method is a very important idea to solve problems, which has a wide range of applications in advanced algebra, due to the limitation of the length of the article and my limited ability, I cannot introduce them one by one. However, through the above induction and arrangement, it is not difficult to find that construction method can simplify and simplify problems in solving mathematical problems, so as to solve problems quickly and give people a sense of winning by surprise. One other thing we have to pay attention to is that constructivism is very useful in solving problems, there is no qualitative method, we need to be flexible and adopt the appropriate method according to the specific problem.

General steps of construction method: *1*)Read the questions carefully and analyze in depth the conditions and conclusions given by the questions.2)According to the given conditions and the characteristics of the conclusion, related mathematical knowledge, to reconstruct the relationship.

3) Thus using the auxiliary elements, find the basic form of the required structure, clear thinking.4) Solve the problem by using the correct construction method and making correct and detailed solutions.

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