Study on Stability Convergence Rate of Chemostat Model of a Kind of Organism

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Abstract — *This paper studies the chemostat model of a kind of organism by using time-delay system stability switching geometric criterion method, discusses the stability and α -stability of the model, and verifies the accuracy and validity of the conclusions by numerical simulation, finally compares the differences and connections between the two stability.*

Keywords —*Delay;Stability; α - stability;Characteristic equation;Stability switching geometriccriterionmethod.*

I. INTRODUCTION

Mathematical methods are widely used in biology and population dynamics. There are also time-delay cases, and in some of the problems, there are phenomenon literature [1-6]with time-delay related parameters.This makes the stability analysis more difficult. The *Lyapunov* function method is usually used, that is, the direction and stability of *Hopf* bifurcation can be given by constructing the central fashion and using the norm method.According to literature [6], *Beretta* and *Kuang* proposed a simple and feasible "geometric criterion method".The method is described below:

Eigen equations for delay differential equations of the following form

$$P_n(\lambda,\tau)+Q_m(\lambda,\tau)\exp(-\lambda\tau)=0 \tag{1}$$

Here $P_n(\lambda,\tau)=\sum_{k=0}^{n} p_k(\tau)\lambda^k$; $Q_m(\lambda,\tau)=\sum_{k=0}^{m} q_k(\tau)\lambda^k$

Where $n,m\in N_0, n>m, p_k(\bullet), q_k(\bullet), P_n, Q_m$ is continuously differentiable with respect to τ .

Let $\lambda=i\omega,(\omega>0)$substitute into equation (1) toget the discriminant function $S_n(\tau)=\tau-\tau_n(\tau),\tau\in I,n\in N_0$ of the stability region. By calculating the zero point τ value of S_n and according to its corresponding discriminant theorem, the stability interval of the equilibrium point corresponding to equation (1) can be obtained.

We generally want to converge to the equilibrium point more quickly, so that we can get the desired result easily by controlling the parameters,In reference [7], the fast convergence problem of a class of time-delay systems is discussed, that is, the α -stability problem of the time-delay dynamic system. The following is the introduction of the α -stability of the time-delay dynamic system:

Consider the characteristic equation of a class of time-delay systems in the following form

$$P(\lambda)+Q(\lambda)e^{-\lambda\tau}=0 \tag{2}$$

$P(\lambda)$ and $Q(\lambda)$ here are both polynomials of λ , and $\partial(P(\lambda))>\partial(Q(\lambda))$ .If all characteristic roots of equation (2) satisfy $\mathrm{Re}(\lambda)<0$, then the zero solution of the system is asymptotically stable.If $\lambda=s-\alpha$,and α are positive real Numbers, equation (2) can be changed to

$$P(s-\alpha)+Q(s-\alpha)e^{\alpha\tau}e^{-s\tau}=0 \tag{3}$$

Write equation (3) as $P(s,\alpha)+Q(s,\tau)e^{-s\tau}=0$ (4)

Here $P(s,\alpha)=P(s\text{-}\alpha)$ is independent of τ, and $Q(s,\tau)=Q(s-\alpha)e^{\alpha\tau}$ is independent of τ.If all the roots satisfy $\mathrm{Re}(s)<0$, then $\mathrm{Re}(\lambda)=\mathrm{Re}(s)-\alpha<-\alpha<0$, for A large positive number α, the solution of the system starting from the equilibrium attachment can quickly converge to the equilibrium point.Therefore, if the equilibrium point is α -stability, it will be asymptotically stability. The reverse is not necessarily true.

In this paper, based on the chemostat model of a class of organisms presented in reference [8], the stability geometric switching method is applied to study the convergence rate of the stability of the chemostat model of a class of organisms with time delay, that is, the stability and α - stability of the model, and the differences and relations between the two kinds of stability are compared.

II. MODEL ESTABLISHMENT

Since the 1960s, many researchers have been working on the chemostat model, which is considered to be a biological model for cultivating microorganisms and studying ecosystems.In order to study the dynamic characteristics of the chemostat model, further understand the problem of microbial culture and study the ecosystem, researchers have done a lot of related work and proposed a lot of related models,in the literature [9], Xiangao Li and Jiaqi Pan et al. studied some dynamic behaviors of the chemostat model by combining it, proposed a new method of studying the model and simply studied the stability of the equilibrium point.In the literature [10], Xiaoyue Li et al. studied the chemostat model with double delay, through which we can more easily master the changes of some microorganisms.In literature [8], the basic model of a kind of organic chemostat is established:

$$\begin{cases} \dot{x}(t)=-(d-p(y(t-\tau)))x(t) \\ \dot{y}(t)=(y_0-y(t))d-\dfrac{1}{\gamma}p(y(t))x(t) \end{cases} \tag{5}$$

Where $x(t)$ represents the concentration of the organism at time t; $y(t)$ represents the concentration of organic protein at time t; $p(y)=\dfrac{cy}{k+y}$ (c and k are normal numbers) represents the growth rate of y; $\gamma>0$ represents the ratio of the organism formed to the medium; $y_0>0$ represents the concentration of organic protein input; $\tau>0$ represents time delay; $d>0$ representsturnover.

III. STABILITY ANALYSIS

Set the equilibrium point of model (5) as $E=(x^*,y^*)$,and then

$$\begin{cases} (d-\dfrac{cy^*}{k+y^*})x^*=0 \\ (y_0-y^*)d-\dfrac{cx^*y^*}{\gamma(k+y^*)}=0 \end{cases} \tag{6}$$

Can solve $x^*=\dfrac{\gamma(cy_0-dy_0-dk)}{c-d}, y^*=\dfrac{dk}{c-d}$

That is $E=(x^*,y^*)=(\frac{\gamma(cy_0-dy_0-dk)}{c-d},\frac{dk}{c-d})$ (7)

Let $x_1(t)=x(t)-x^*, y_1(t)=y(t)-y^*$,then, the linear approximation system obtained by linearizing model (5) at equilibrium point $E=(x^*,y^*)$ is:

$$\begin{cases} \dot{x}_1(t)=(n-d)x(t)+my_1(t-\tau) \\ \dot{u}_2(t)=-\frac{n}{\gamma}x_1(t)-(\frac{m}{\gamma}+d)y_1(t) \end{cases} \tag{8}$$

in which $m=\frac{kcx^*}{(k+y^*)^2}, n=\frac{cy^*}{k+y^*}$.

The characteristic equation of equation (8) is

$$\lambda^2+(\frac{m}{\gamma}-n+2d)\lambda-(n-d)(\frac{m}{\gamma}+d)+\frac{mn}{\gamma}e^{-\lambda\tau}=0 \tag{9}$$

Let $\frac{m}{\gamma}+d=A, n-d=B, \frac{mn}{\gamma}=K$,Then the characteristic equation (9) can be rewritten as

$$\lambda^2+(A-B)\lambda-AB+Ke^{-\lambda\tau}=0 \tag{10}$$

When $\tau=0$, the literature [8] has proved that $\mathrm{Re}(\lambda)<0$, that is, the model is stable at the equilibrium point.Next, observe whether the real part of some characteristic roots of characteristic equation (10) will increase to zero or even become positive as the delay value τ increases.

When $\tau>0$, the literature [8] has also proved that the model is stable at the equilibrium point, but its proof process is relatively complex. Now it is proved by using the time-delay system stability switching geometric criterion method.

Take $\lambda=i\omega$, substitute it into equation (10), and you get

$$-\omega^2+(A-B)\omega i-AB+K(\cos\omega\tau-i\sin\omega\tau)=0 \tag{11}$$

Separate the real part from the imaginary part and you get

$$\begin{cases} \omega^2+AB-K\cos\omega\tau=0 \\ (A-B)\omega-K\sin\omega\tau=0 \end{cases} \tag{12}$$

Equation $F(\omega,\tau)=\omega^4+(A^2+B^2)\omega^2+A^2B_1^{\ 2}-K^2=0$ (13)

can be obtained from equation (12).

You get $\omega^2 = \dfrac{-(A^2+B^2) \pm \sqrt{(A^2+B^2)^2 - 4(A^2B^2 - K^2)}}{2}$ (14)

That is $\omega = \sqrt{\dfrac{-(A^2+B^2) + \sqrt{(A^2+B^2)^2 - 4(A^2B^2 - K^2)}}{2}}$ (15)

From literature[11],you can get function $\sin\theta = \dfrac{(A+B)\omega}{K}, \cos\theta = \dfrac{\omega^2 - AB}{K}$ (16)

So $\tan\theta = \dfrac{(A+B)\omega}{\omega^2 - AB}$ (17)

From literature[11],youknow $S_j = \tau - \tau_j = \tau - \dfrac{\theta + j2\pi}{2\omega}, j = 0,1,2\cdots\cdots$ (18)

Let's take $S_j = 0,$, find the zero and call it τ_j, and the derivative of S_j with respect to τ is $\dfrac{d(S_j)}{d\tau} = 1 > 0$, and of course S_j is A monotone increasing function, so S_j has and only has one zero, not even zero,according to literature [11], the zero point of S_0 is the first and unique stability switching point of model (5).

Therefore $S_0 = \tau - \tau_0 = \tau - \dfrac{\theta}{2\omega} = 0$,that is $\tau_0 = \dfrac{\theta}{2\omega}$ produces the unique stability switching point of equilibrium for the chemostat model of a class of organisms, so we can get that the stability interval of this model is $\left[0, \dfrac{\theta}{2\omega}\right)$.

IV. NUMERICAL SIMULATION OF MODEL STABILITY

This section will give a specific example to verify the feasibility of the above theoretical analysis results through numerical simulation.

The parameter selected in this article is $y_0 = 0.75, c = 0.6, d = 0.45, \gamma = 2, k = 0.1$. It's easy to figure out with mathematical software that the equilibrium point is $E = (x^*, y^*) = (0.92, 0.3); \omega = 0.15167;$

$\tau_0 = 8.373$,Therefore, under this parameter, model (5) is asymptotically stable when $\tau \in [0, \tau_0)$; when $\tau = \tau_0$, the system produces *Hopf* branch, that is the periodic solution; when $\tau \in (\tau_0, \infty)$, it is unstable.

When $\tau = 7.5 < \tau_0$ is selected, the system is asymptotically uniform and stable at the equilibrium point, as shown in figure 1;Similarly, it can be simulated numerically,When $\tau = 8.373 = \tau_0$, the system produces *Hopf* branch, as shown in figure 2;When $\tau = 15 > \tau_0$, the system is unstable at the equilibrium point, as shown in figure 3.Numerical simulation shows the changing process of the system

from stable to unstable.

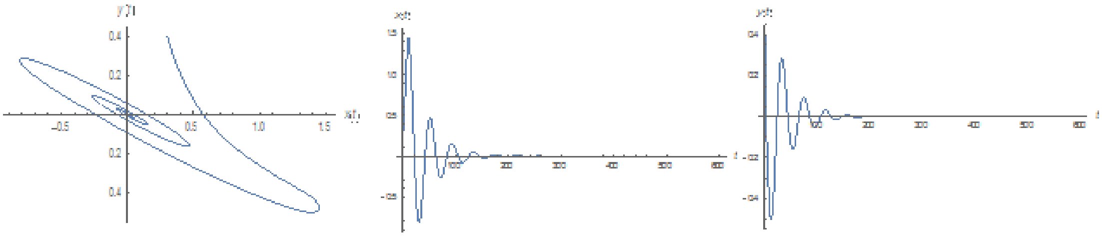

Fig. 1 Phase diagram and time history diagram of $x(t)$、$y(t)$ when $\tau = 7.5 < \tau_0$

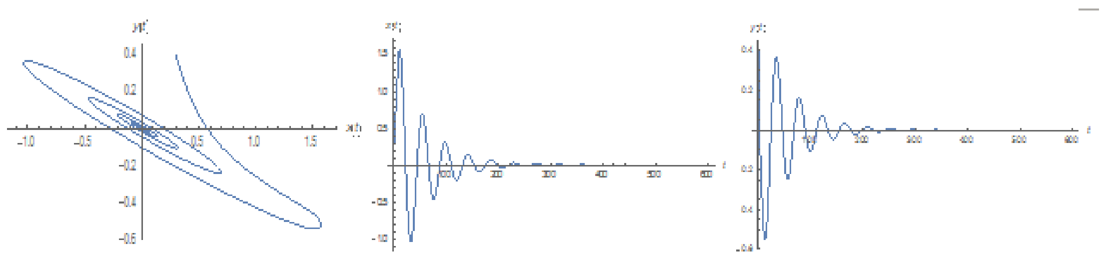

Fig. 2 Phase diagram and time history diagram of $x(t)$、$y(t)$ when $\tau = 8.373 = \tau_0$

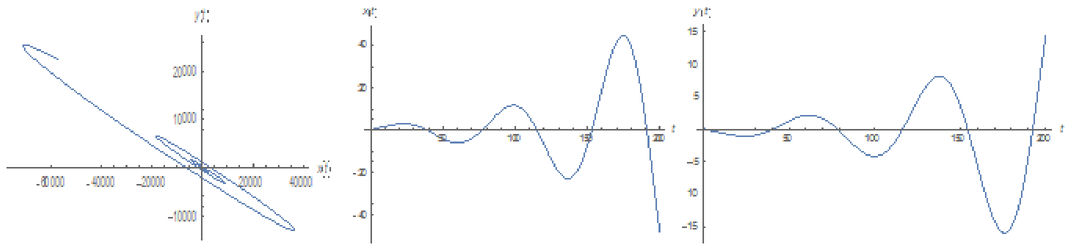

Fig. 3 Phase diagram and time history diagram of $x(t)$、$y(t)$ when $\tau = 15 > \tau_0$

V. α -STABILITY STUDY

According to literature [7], we have learned the definition of α - stability of dynamical systems with time delay and other basic knowledge.In this paper, we have also made a brief introduction,we know that α - stability converges faster than stability at the equilibrium point under certain conditions, so it is necessary to study α - stability.In this section, by introducing a α factor, the time-delay system with parameters unrelated to time delay is transformed into time-delay system with parameters related to time delay. Then,the chemostat model of a kind of organism is analyzed and discussed by using the formula $S_j = \tau - \tau_j$ and its conclusion in the papers of *Beretta*

and *Kuang* , that is, the α - stability problem.

In section 3, we obtain the characteristic equation of a kind of organic chemostat model, which is equation

$$(10) \quad \lambda^2 + (A-B)\lambda - AB + Ke^{-\lambda\tau} = 0 .$$

Let $\lambda = s - \alpha$, α is a constant greater than zero, the above equation can be reduced to

$$D(\lambda) = D(s,\tau) = (s-\alpha)^2 + (A-B)(s-\alpha) - AB + Ke^{-\tau(s-\alpha)} = 0 \tag{19}$$

Equation (19) can be denoted as $D(s,\tau)=s^2+a_1s+b_1+c_1e^{-\tau s}=0$ (20)

Where $a_1=A-B-2\alpha;b_1=\alpha^2-(A-B)\alpha-AB;c_1=Ke^{\tau\alpha}$.

When $\tau=0$, equation (10) is converted to $\lambda^2+(A-B)\lambda-AB+K=0$, and if it is α -stability,from literature[7], we can get $A-B>2\alpha;K-AB>\alpha^2$.

When $\tau>0$,let $s=i\omega$,substituted into equation (20) to get

$$-\omega^2+ia_1\omega+b_1+c_1(\cos\omega\tau-i\sin\omega\tau)=0 \tag{21}$$

The above equation separates the real part from the imaginary part,we get:

$$\begin{cases}-\omega^2+b_1+c_1\cos\omega\tau=0\\ a_1\omega-c_1\sin\omega\tau=0\end{cases} \tag{22}$$

From above,we get $F(\omega,\tau)=\omega^4+(a_1{}^2-2b_1)\omega^2+b_1{}^2-c_1{}^2=0$ (23)

So $\omega^2=\dfrac{-a_1{}^2+2b_1+\sqrt{a_1{}^4-4a_1{}^2b_1+4c_1{}^2}}{2}$ (24)

That is $\omega=\sqrt{\dfrac{-a_1{}^2+2b_1+\sqrt{a_1{}^4-4a_1{}^2b_1+4c_1{}^2}}{2}}$ (25)

From literature [11], we get function $\sin\theta=\omega\dfrac{a_1}{c_1},\cos\theta=\dfrac{\omega^2-b_1}{c_1}$ (26)

So $\tan\theta=\dfrac{a_1\omega}{\omega^2-b_1}$ (27)

From the literature[11] again know $R_n=\tau-\tau_n=\tau-\dfrac{\theta+n2\pi}{2\omega},n=0,1,2\cdots\cdots$ (28)

At the same time there is $R(\tau)=\mathrm{sgn}\{\dfrac{d\,\mathrm{Re}(\lambda)}{d\tau}\}=\mathrm{sgn}\{\dfrac{dR_n}{d\tau}\}$ (29)

Let $R_n=0$, , the zero point is denoted as τ_n, and the derivative of R_n with respect to τ is denoted as $\dfrac{d(R_n)}{d\tau}=1>0$. Obviously, R_n is a monotone increasing function, so R_n has and only has one zero point, non-even number of zero points. According to text [11], the zero point of R_0 is the first and unique stability switching point of model (5), denoted as τ_{01}.

VI. NUMERICAL SIMULATION OF MODEL α - STABILITY

According to the relevant parameters in the numerical simulation of model stability in section 4, and let the factor is $\alpha=0.1$, we study the convergence rate of the model at this time, that is α - stability.

We can calculate $\tau_{01}=6.59326$ through mathematical software,that is the model is α - stability when $\tau<\tau_{01}=6.59326$

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When $\tau = 5 < \tau_{01}$ is selected, the model is α - stability at the equilibrium point, as shown in figure 4;Similarly, when $\tau_{01} < \tau = 7.5 < \tau_0$, the model is asymptotically stable at the equilibrium point, as shown in figure. 5;By comparing figure 4 and figure 5, it is clear that the convergence rate of figure 4 is faster than that of figure 5. Then, with the increase of τ value, the model will gradually become unstable, as shown in figure 3.

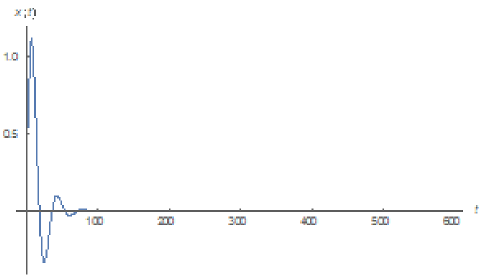

Fig.4Time history diagram of $x(t)$、$y(t)$ when $\tau = 5 < \tau_{01}$

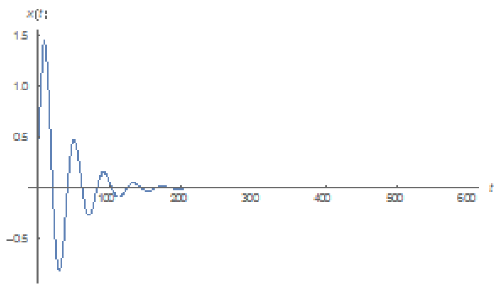

Fig.5 Time history diagram of $x(t)$、$y(t)$ when $\tau_{01} < \tau = 7.5 < \tau_0$

VII. CONCLUSION

In this paper, the convergence rate of the local stability of a type of organic chemostat model (5) is studied.Firstly, the stability of model (5) is studied by using the stability geometric switching method,compared with the previous method of constructing central manifolds and using norm, namely *Lyapunov* function method, the method used in this paper is simpler.Therefore, the idea of stability geometric switching has great advantages in the study of local stability of time-delay dynamical systems.Secondly, the α -stability of model (5) is studied by using the time-delay dynamical system stability geometric switching method, that is, by introducing the factor α and controlling its size, the goal of fast convergence can be achieved.At last, the scientific nature of the theory is verified by numerical simulation with mathematical software on the basis of theoretical analysis.

The research results of this paper can be applied to real life. Different parameters are selected for different situations and substituted into the results of this paper to get τ_0 and τ_{01}, so as to carry out detection experiments.The preceding arguments set the stage for further research, and there are many unexplored theories to explore.We can also assign different values to α to obtain different τ_{01}, so as to further study the convergence rate of the model and select the optimal one according to the actual situation, which is conducive to better control of the system.

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REFERENCES

[1] F. Brauer A. C. Soudack. On Constant Effort Harvesting and Stocking in a class of Predator-prey[J]. J. Math. Biol, 1982, 9: 247-252.
[2] Y. Kuang. Delay Differential Equation with Applications to Population Dynamics[M]. New York: Academic Press, 1993.
[3] Lansun Chen. Mathematical ecological model and research methods [M]. Beijing: science press, 1988.
[4] Z.Xiong.Study on the prameters modelin the type II functional response periodic of three populations[J]. J.Math Biol, 1998, 13(1): 38-42.
[5] K.L.Cooke P. V. Diressche X. Zou. International maturation delay and nonlinear birth in population and epidemic models[J]. J. Math. Biol, 1999, 39: 332-352.
[6] E. Beretta Y. Kuang. Geometric stability switches criteria in delay differential systems with delay dependent parameters [J]. SIAM. J. Math. Anal., 2002, 33: 1144-1165.
[7] Chengkuan Di. α -stability analysis of time-delay dynamical systems [J]. Journal of nanjing university of technology (natural science edition),2008,9(03):1-6.
[8] Haiyun Bai，Yanhui Zhal．Hopf bifurcation analysis for the model of the Chemostat with one species of organism[J]．Abstract and Applied Analysis，2013，2013：1—7．
[9] Xiangao Li, Jiaqi Pan. $Hopf$ Bifurcation Analysis of Some Modified Chemostat Models[J]. Northeast Math，1998，14(4)：392—400．
[10] Xiaoyue Li, Meihua Qian,Jianping Yang. $Hopf$ Bifurcations of a Chemostat System with Bi-parameters[J].Northeast Math，2004，20(2)：167-174．
[11] BERETTA E,KUANG Y.Geometric stability switches criteria in delay differential systems with delay dependent parameters[J].SIAM J Math Anal,2002,33:1144-1165.