Common Fixed Point Theorems using Common E.A. Like Property in Fuzzy Metric Space

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Abstract

In the present paper, we prove common fixed point theorems in fuzzy metric spaces using common E.A. like property with weakly compatible mappings. Our results improve the results of [2].

Keyword:

Fuzzy metric space, common E.A. like property, weakly compatible mappings.

I. INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh [1] in 1965. The fuzzy sets has been developed by many researchers in different spaces and introduced new theories like fuzzy topology role, fuzzy normed space, fuzzy metric space and so on. Kamosil and Michalek [3] in 1975 have introduced the concept of fuzzy metric space, the continuous triangular norm defined by Schweizer and Sklar [5].

Different mappings have been used by various authors to obtain fixed point theorems in FMS. There is vast literature in fixed point theory in fuzzy metric space. Researcher used different types of commuting mappings to prove fixed point theorems under different contractive conditions. Weak compatibility is one of the weaker forms of the commuting mappings. Many researchers use this concept to prove the existence of unique common fixed point in fuzzy metric space under contractive conditions. On the other hand, Wadhwa et al. [6] introduced the notion of common E.A. like property and proved some common fixed point theorems in fuzzy metric spaces. Recently, Hassan and Alla [2] proved common fixed point theorems for mappings satisfying weak compatibility and semi-compatibility with reciprocal continuity in fuzzy metric space.

In this paper, we prove common fixed point theorems in fuzzy metric spaces using common E.A. like property with weakly compatible mappings. Our results improve the results of [2]

II. PRELIMINARIES

Definition 2.1[5]: A binary operation *: $[0, 1] \times [0, 1] \to [0, 1]$ is continuous t- norm if * satisfies the following conditions $\forall a, b, c, d \in [0,1]$:

(i) * is commutative and associative;
(ii) * is continuous;
(iii) $a * 1 = a$;
(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$.

Definition 2.2[1]: The 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions: $\forall x, y, z \in X$, $t, s > 0$;

(1) $M(x, y, t) = 0$;
(2) $M(x, y, t) = 1$ iff $x = y$;
(3) $M(x, y, t) = M(y, x, t)$;
(4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$;
(5) $M(x, y, \cdot): [0,\infty) \to [0, 1]$ is left continuous;
(6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$.

Lemma 2.3[2]: For every $x, y \in X$, the mapping $M(x, y, t)$ is increasing on $(0,\infty)$.

Definition 2.4[3]: The 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2\times(0,\infty)$ satisfying the following conditions: $\forall$x, y, z $\in$X, t, s > 0;

(GV - 1) M(x, y, t) > 0;
(GV - 2) M(x, y, t) = 1 iff x = y;
(GV - 3) M(x, y, t) = M(y, x, t);
(GV - 4) M(x, y, t)*M(y, z, s) $\leq$ M(x, z, t + s);
(GV - 5) M(x, y, ·): $[0,\infty) \to [0, 1]$ is continuous.

Example 2.5: Let (X, d) be a metric space. Define $a*b=ab$ and for all x, y$\in X$ and t>0,

$$M(x, y, t) = \frac{t}{t+d(x,y)},$$

Then $(X, M, *)$ is fuzzy metric space.

Definition 2.6[2]: Two self maps$A$ and B of a fuzzy metric space $(X,,*)$ is said to be weak-compatible if they commute at their coincidence points, i.e $Ax = Bx$ implies $ABx = BAx$

Definition 2.7[2]: A pair $(A,)$ of self maps of a fuzzy metric space $(X,,*)$ is said to be semi-compatible if $\lim_{n\to\infty}ASx_n = Sx$, whenever there exists a sequence x_n in X such that $\lim_{n\to\infty}Ax_n = \lim_{n\to\infty}Sx_n = x$, for some x$\in$X.

Definition 2.8[2]: A pair $(A,)$ of self maps of a fuzzy metric space $(X,,*)$ is said to be reciprocal continuous if $\lim_{n\to\infty}ASx_n=Ax$ and $\lim_{n\to\infty}SAx_n=Sx$ whenever there exists a sequence $x_n\in X$ such that $\lim_{n\to\infty}Ax_n= \lim_{n\to\infty}Sx_n= x$ for some $x\in X$

Definition 2.9 [6]: Let A, B, S and T be self maps of a fuzzy metric space $(X, M,*)$, then the pairs (A, S) and (B, T) said to satisfy common E.A. Like property if there exist two sequences x_n and y_n in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$, for some $z \in S(X) \cap T(X)$ or $z \in A(X) \cap B(X)$.

Lemma 2.10[2, 4]: If there exists k$\in$(0,1) such that $M(x, y, kt) \geq M(x, y, t)$ for all x, y$\in X$ and t$\in(0,\infty)$ then x=y.

III. MAIN RESULT

Theorem 3.1: Let (X, M, *) be a fuzzy metric space where * is continuous t-norm and satisfies $t * t \geq t$ for all t$\in$[0, 1]. Let A, B, S and T be self mappings on a fuzzy metric space satisfying the following conditions:

(3.1.1) $\forall$ x, y$\in$X, t>0,

$$M(Ax, By, t) \geq F(M(Sx, Ty, t)),$$

where $F: [0,1]\to[0,1]$ is a continuous function such that $F(0) = 0, F(1) = 1$ and $F(a) > a$ for each $0 < a < 1$,

(3.1.2) pairs (A, S) and (B, T) satisfy common E.A. like property,

(3.1.3) pairs (A, S) and (B, T) are weakly compatible,

then A, B, S and T have a unique common fixed point in X.

Proof: Since (A, S) and (B, T) satisfy common E. A. Like property therefore there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty}Ax_n = \lim_{n\to\infty}Sx_n = \lim_{n\to\infty}Ty_n = \lim_{n\to\infty}By_n = z$$

where $z \in S(X)\cap T(X)$ or $z \in A(X)\cap B(X)$.

Suppose $z \in S(X)\cap T(X)$, now we have $\lim_{n\to\infty}Ax_n = z \in S(X)$ then z = Su for some u $\in$ X.

Now, we claim that Au = Su. From (3.1.1) we have,

$$M(Au, By_n, t) \geq F(M(Su, Ty_n, t)),$$

Taking limit $n \to \infty$, we get

$$M(Au, By_n, t) \geq F\big(M(z, z, t)\big) = F(1) = 1,$$

$$M(Au, z, t) \geq 1,$$

it implies that, $Au = z = Su$.

Since the pair (A, S) is weak compatible, therefore Az = ASu = SAu = Sz.

Again, $\lim_{n\to\infty} By_n = z \in T(X)$ then z= Tv for some v∈ X.

Now, we claim that Tv = Bv. From (3.1.1) we have,

$$M(Ax_n, Bv, t) \geq F(M(Sx_n, Tv, t))$$

Taking limit n → ∞, we get

$$M(z, Bv, t) \geq F\big(M(z, z, t)\big) = F(1) = 1,$$

which implies that $Bv = z = Tv$.

Since the pair (B, T) is weak compatible, therefore Tz = TBv = BTv = Bz.

Now, we show that Az = z. From (3.1.1) we have,

$$M(Az, By_n, t) \geq F(M(Sz, Ty_n, t))$$

Taking limit n → ∞, we get

$$M(Az, z, t) \geq F\big(M(Az, z, t)\big) > M(Az, z, t),$$
$$M(Az, z, t) > M(Az, z, t),$$

a contradiction. Therefore, $Az = z$.

Now we show that Bz = z. From (3.1.1) we have,

$$M(Ax_n, Bz, t) \geq F(M(Sx_n, Tz, t))$$

Taking limit n → ∞, we get

$$M(z, Bz, t) \geq F(M(z, Bz, t)) > M(z, Bz, t),$$
$$M(z, Bz, t) > M(z, Bz, t),$$

a contradiction. Therefore, $Bz = z$. Hence, $Az = Sz = Bz = Tz = z$.

Thus z is common fixed point of A, B, S and T. The uniqueness follows from contractive condition. This completes the proof of the theorem.

Remark 3.2: Theorem 3.1 never requires the completeness of the space and continuity of the involved mappings. We replaced semi-compatible mapping by common E.A. like property and improved the result of [2].

Now, we prove another common fixed point theorem with different contractive condition:

Theorem 3.3: Let (X, M, *) be a fuzzy metric space where * is continuous t-norm and satisfies t * t ≥ t for all t∈[0, 1]. Let A, B, S and T be self mappings on a fuzzy metric space satisfying the following conditions:

(3.3.1) ∀ x, y∈X, t>0,

$$M(Ax, By, t) \geq F(\min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, t)\}),$$

where F: [0,1]→[0,1] is a continuous function such that $F(0) = 0, F(1) = 1$ and $F(a) > a$ for each $0 < a < 1$,

(3.3.2) pairs (A, S) and (B, T) satisfy common E.A. like property,

(3.3.3) pairs (A, S) and (B, T) are weakly compatible,

then A, B, S and T have a unique common fixed point in X.

Proof: Since (A, S) and (B, T) satisfy common E. A. like property therefore there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$$

where z ∈S(X)∩T(X) or z ∈A(X)∩B(X).

Suppose z ∈S(X)∩T(X), now we have $\lim_{n\to\infty} Ax_n = z \in S(X)$ then z = Su for some u ∈ X.

Now, we claim that Au = Su. From (3.3.1) we have,

$$M(Au, By_n, t) \geq F(\min\{M(Su, Ty_n, t), M(Au, Su, t), M(By_n, Ty_n, t), M(Au, Ty_n, t)\}),$$

Taking limit $n \to \infty$, we get

$$M(Au, z, t) \geq F(\min\{M(z, z, t), M(Au, z, t), M(z, z, t), M(Au, z, t)\}),$$
$$M(Au, z, t) \geq F(\min\{1, M(Au, z, t), 1, M(Au, z, t)\}),$$
$$M(Au, z, t) \geq F\big(M(Au, z, t)\big) > M(Au, z, t),$$

a contradiction. Therefore, $Au = z = Su$.

Since the pair (A, S) is weak compatible, therefore Az = ASu = SAu = Sz.

Again, $\lim_{n\to\infty} By_n = z \in T(X)$ then z= Tv for some $v \in X$.

Similarly, from condition (3.3.1), we can prove $Bv = z = Tv$.

Since the pair (B, T) is weak compatible, therefore Tz = TBv = BTv = Bz.

Now we show that Az = z. From (3.3.1) we have,

$$M(Az, By_n, t) \geq F(\min\{M(Sz, Ty_n, t), M(Az, Sz, t), M(By_n, Ty_n, t), M(Az, Ty_n, t)\}),$$

Taking limit $n \to \infty$, we get

$$M(Az, z, t) \geq F(\min\{M(Az, z, t), M(Az, Az, t), M(z, z, t), M(Az, z, t)\}),$$

$$M(Az, z, t) \geq F(\min\{M(Az, z, t), 1, 1, M(Az, z, t)\}),$$
$$M(Az, z, t) \geq F\big(M(Az, z, t)\big) > M(Az, z, t),$$

a contradiction. Therefore, $Az = z$. Similarly, from condition (3.3.1), we can prove $Bz = z$.

Hence, $Az = Sz = Bz = Tz = z$. Thus z is common fixed point of A, B, S and T. Uniqueness of fixed point can be easily verify by contractive condition. This completes the proof of the theorem.

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